Grupo de Mecânica
Aplicada e Computacional

Federal University
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The Element-Free Galerkin Method applied on Polymeric Foams

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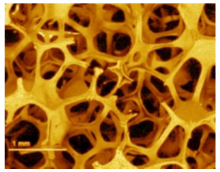
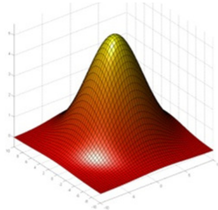
Overview

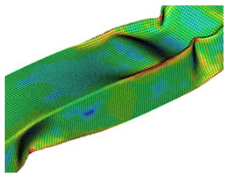
- **Celular Solids**
  - Introduction
  - Characteristics
  - Mechanical Properties
  - Models examples
  - Methodology



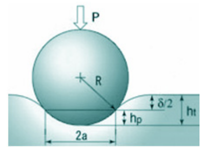
- **The Element-Free Galerkin Method**
  - Moved List Square Approximation
  - Weight Function definition
  - Imposition of Essential Boundary Conditions



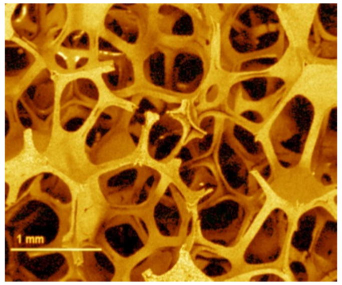
- **Finite Strain Elastoplastic formulation**
  - Kinematics of deformation
  - The concept of conjugated pairs
  - The Elastoplastic model
  - Strong, Weak and Incremental formulations
  - Solution Procedure
  - Examples (FEM)



- **Unilateral Contact and Friction Formulation**
  - Problem definition
  - Imposition of the Contact and Friction terms
  - General Algorithm
  - Examples (EFG)



- **Conclusions**



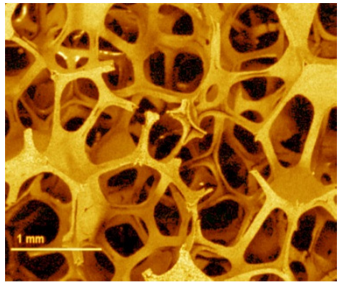
Cellular solids

Introduction

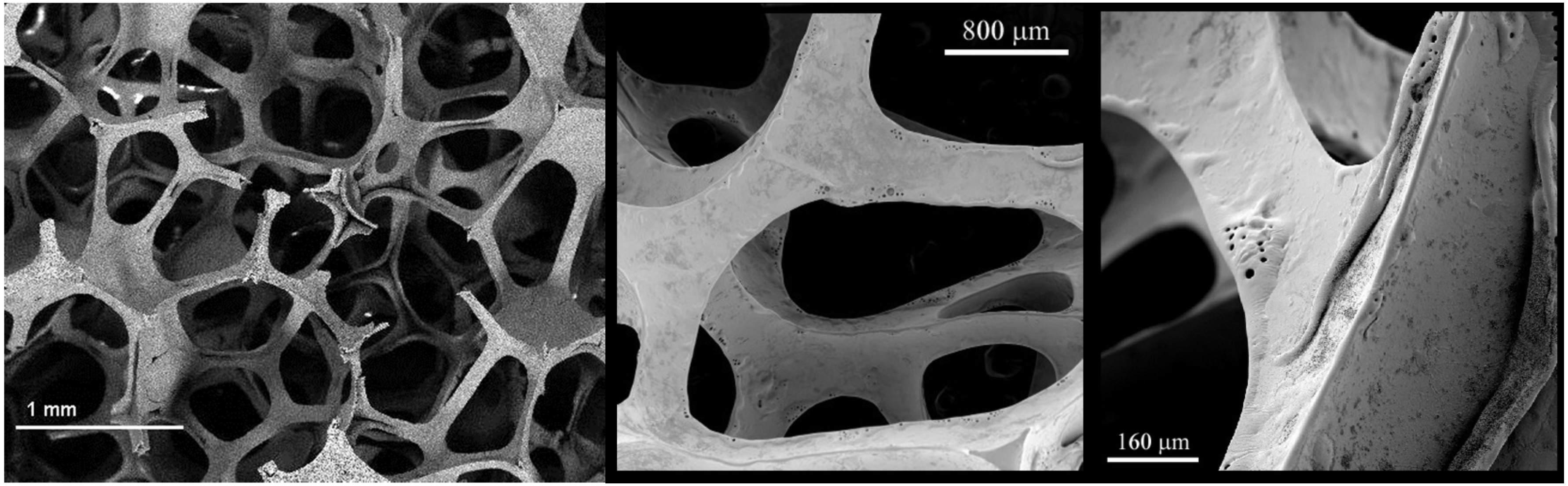
- Typically used in energy absorption structures.

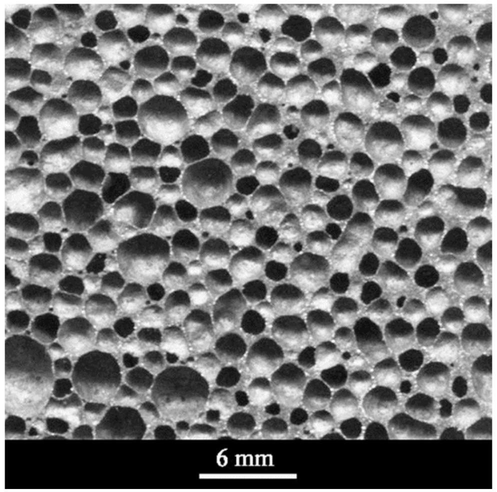
- Applications areas:
  - Automotive/Transport industry;
  - Aerospace industry;
  - Packing industry;
  - Construction industry.

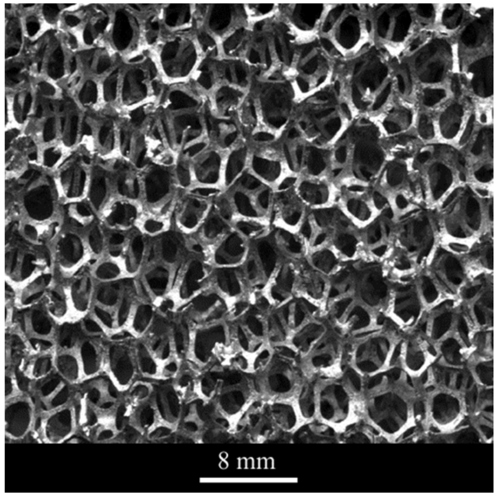
- Mechanical X low density.

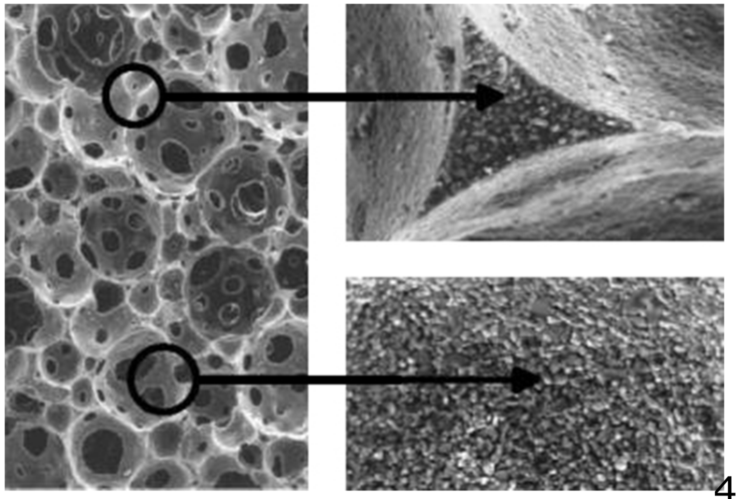


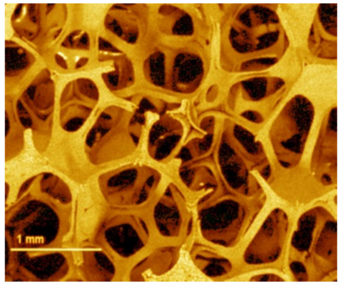
Cellular solids











Cellular solids

- **Relative density**

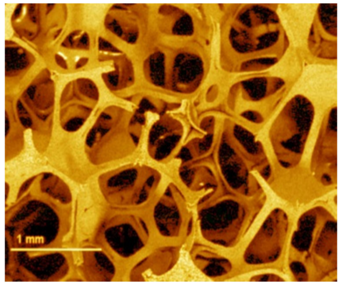
$$\rho^* = \rho/\rho_m$$

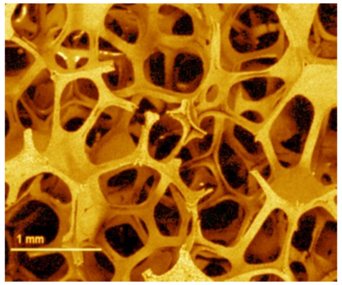
  - Very low = 0,001;
  - Conventional = 0,05 to 0,20;
  - Transition value 0,3 treated as a solid with isolated pores.

- **Types**
  - Polymeric
  - Metallic (aluminum, cooper, nickel, titanium and zinc)
  - Ceramic (carbon)
  - Natural (wood, cork and coral structures)

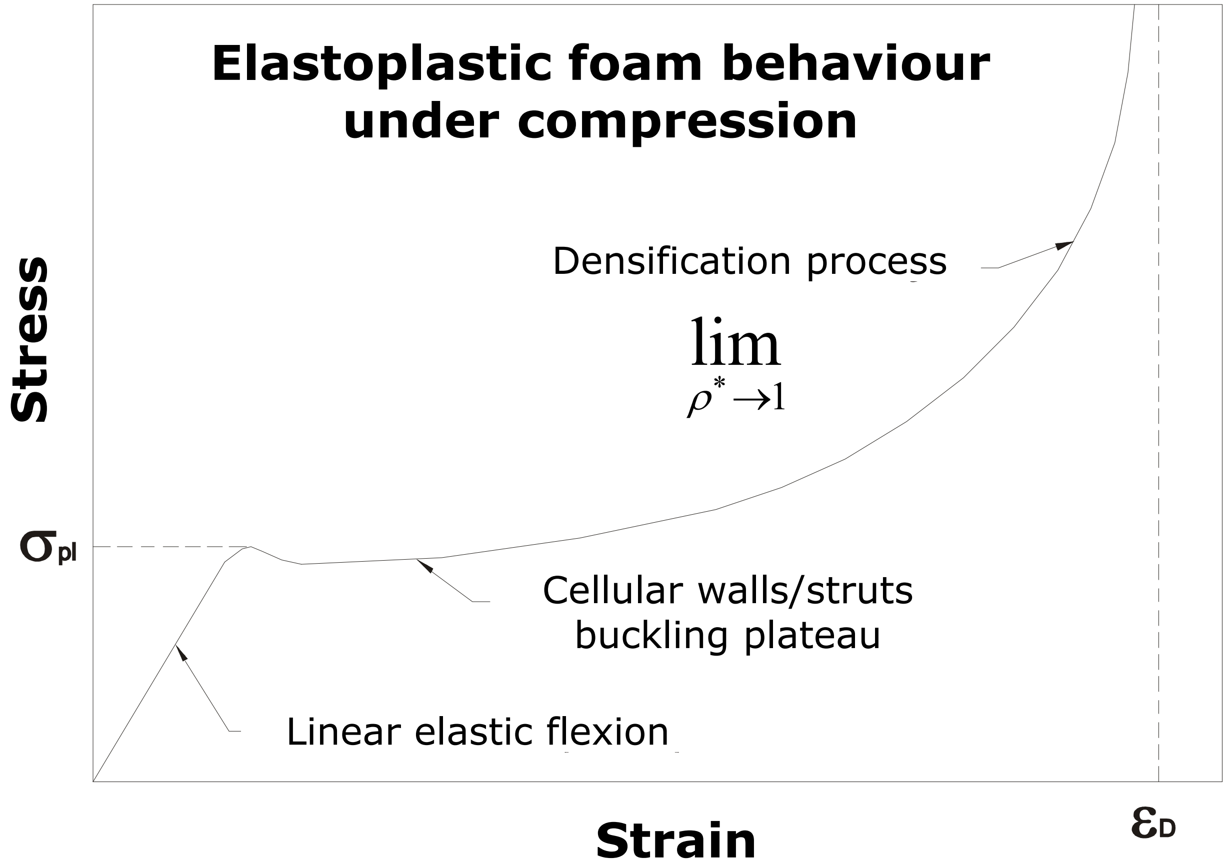
Mechanical Properties

GIBSON & ASHBY(1997)

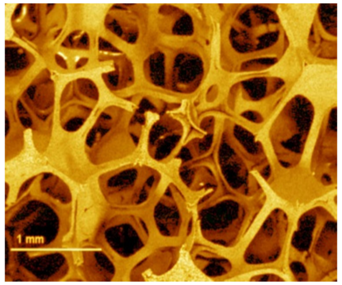




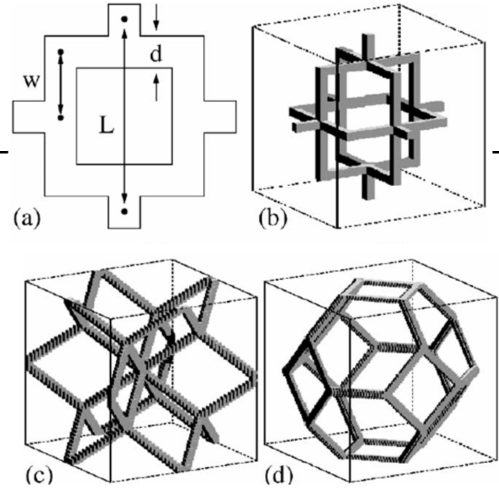
Cellular solids



- **Mechanical Properties**
  GIBSON & ASHBY(1997)

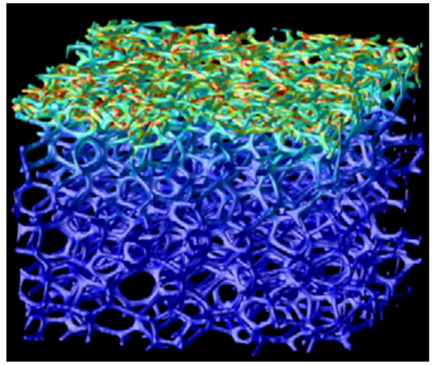


Cellular solids

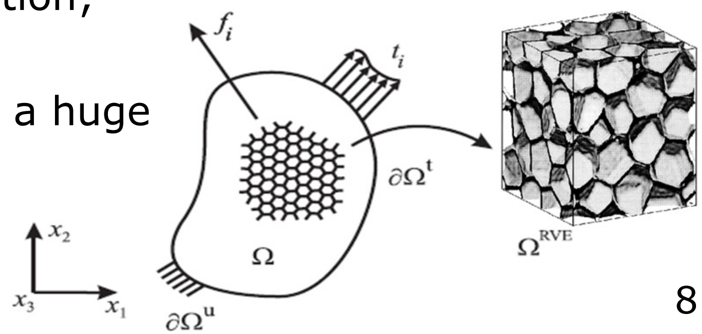


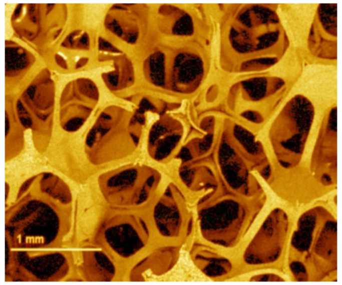
- **Modeling Approaches**

  - Periodical Models (GIBSON & ASHBY)
    - Dependence of the geometrical idealization;
    - Considers the local cell characteristics;
    - Limited applications;
    - Isotropy and cellular interaction, in the majority of the cases, are not periodic phenomenon.



  - Random models (ROBERTS,GARBOCZIA e BRYDON)
    - *Voronoi* tessellation and *Gaussian* random field;
    - Problems to define the RVE and the coordinate number;
    - Problems at the micro tomography correlation;
    - Border effects;
    - Self-contact densification process involves a huge computational cost;
    - Explicit solver.





Cellular solids

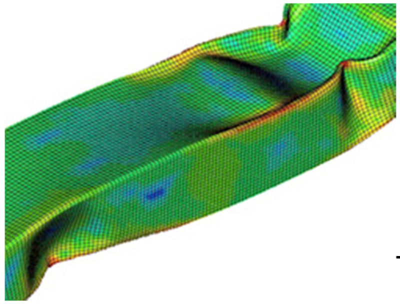
- **Elastoplastic model for polymeric foams**

- **Methodology**

  - Finite Stains Algorithm;
  - *Total Lagrange* Description;
  - *Rotated Kirchoff* stress*;*
  - *Hencky* Logarithmic strain measure;
  - Volumetric hardening Law;

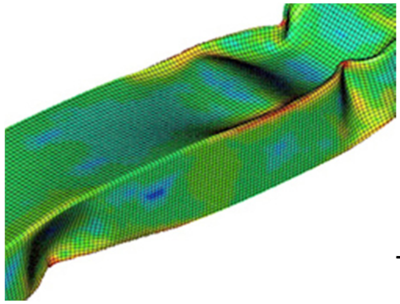
  - Finite Element Method - FEM;
  - Element Free Galerkin Method - EFG;

  - Contact formulation (*Signorini* hypothesis);
  - Friction formulation (regularized *Coulomb's* law);



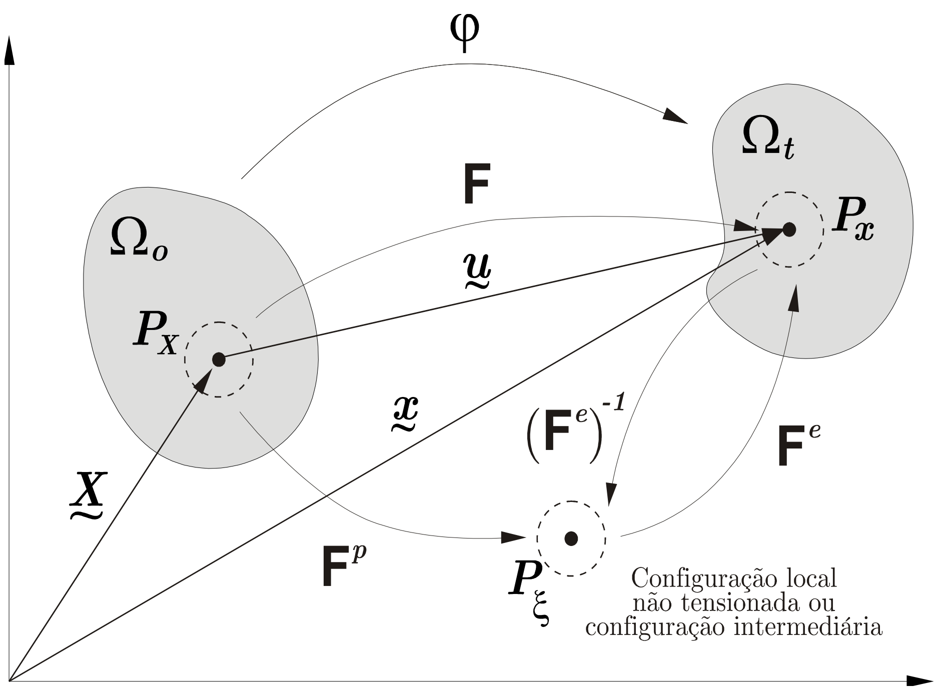
Finite Strain Elastoplastic Formulation

- **Required formulations**
  - Stress/Strain elastic relation – an elastic law;
  - Yield function - indicating the stress level to start the plastic flow;
  - Stress/Strain plastic relation – material hardening/softening law;



Finite Strain Elastoplastic Formulation

- **Kinematics of deformation**



- Movement, strain gradient and the multiplicative decomposition

$$\varphi(\vec{X},t) = \vec{x} = \vec{X} + \vec{u}$$

$$\mathbf{F} = \nabla_{\vec{X}}\varphi(\vec{X},t) = \frac{\partial \vec{x}}{\partial \vec{X}}$$

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$$

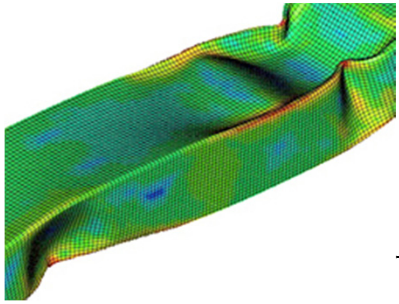
- Polar decomposition, *Cauchy-Green* tensor and the log strain measure

$$\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p \qquad \mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e$$

$$\mathbf{U}^e = \sqrt{\mathbf{C}^e}$$

$$\mathbf{C}^e = \mathbf{F}^{e^T}\mathbf{F}^e$$

$$\mathbf{E}^e = \ln\left(\mathbf{U}^e\right)$$



Finite Strain Elastoplastic Formulation

- ***Hill's*** **Principle: work rate invariability**

$$\dot{\mathcal{W}} = \frac{1}{\rho}\boldsymbol{\sigma}\cdot\mathbf{D} = \frac{1}{\rho_o}\boldsymbol{\tau}\cdot\mathbf{D} = \frac{1}{\rho_o}\mathbf{P}\cdot\dot{\mathbf{F}} = \frac{1}{2\rho_o}\mathbf{S}\cdot\dot{\mathbf{C}} = \frac{1}{\rho_o}\bar{\boldsymbol{\tau}}\cdot\dot{\mathbf{E}}^e,$$

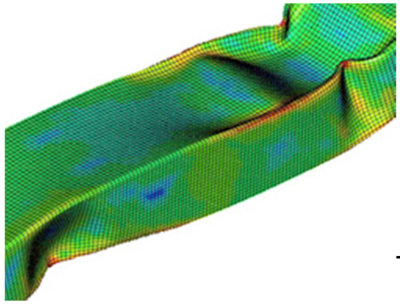
- **Spectral decomposition**

$$\mathbf{C}^e = \mathbf{F}^{e^T}\mathbf{F}^e$$

$$\mathbf{C}^e = \sum_{i=1}^{3}\lambda_i(\vec{l}_i \otimes \vec{l}_i)$$

$$\mathbf{U}^e = \sum_{i=1}^{3}\sqrt{\lambda_i}(\vec{l}_i \otimes \vec{l}_i)$$

$$\mathbf{E}^e = \frac{1}{2}\sum_{i=1}^{3}\ln(\lambda_i)(\vec{l}_i \otimes \vec{l}_i)$$



Finite Strain Elastoplastic Formulation

- ***Kirchoff rotated* stress**

$$\bar{\boldsymbol{\tau}} = (\mathbf{R}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\tau}\ (\mathbf{R}^{\mathrm{e}}) \qquad \text{where} \qquad \boldsymbol{\tau} = \det(\mathbf{F})\boldsymbol{\sigma}$$

- **Hyperelastic *Hencky's* model**

$$\bar{\boldsymbol{\tau}} = \mathbb{D}(\rho^*)\mathbf{E}^e$$

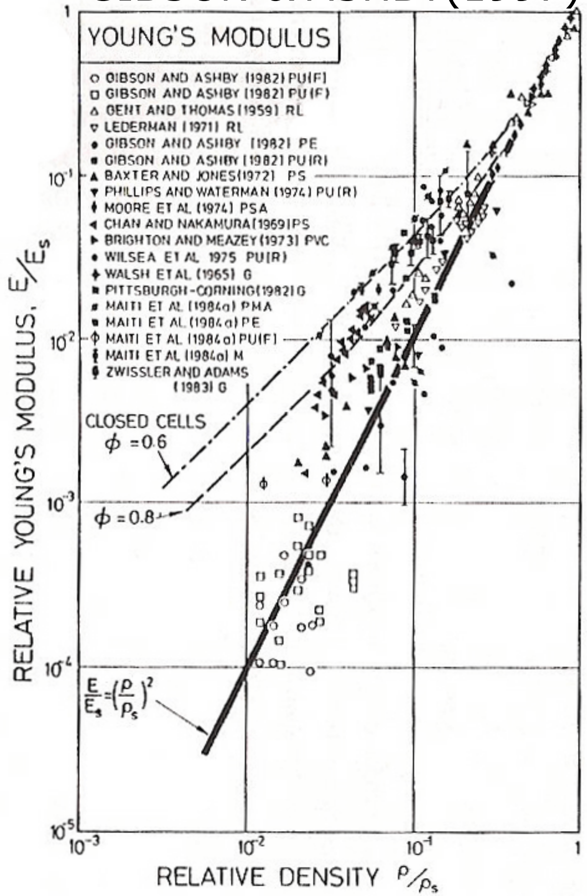
$$\mathbb{D}(\rho^*) = 2\mu(\rho^*)\mathbb{I} + \left(K(\rho^*) - \frac{2}{3}\mu(\rho^*)\right)(\mathbf{I}\otimes\mathbf{I})$$

$$\rho_o^* = \det[\mathbf{F}]\rho^*$$

$$\mathbb{I}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$(\mathbf{I}\otimes\mathbf{I})_{ijkl} = \delta_{ij}\delta_{kl}$$

GIBSON & ASHBY(1997)

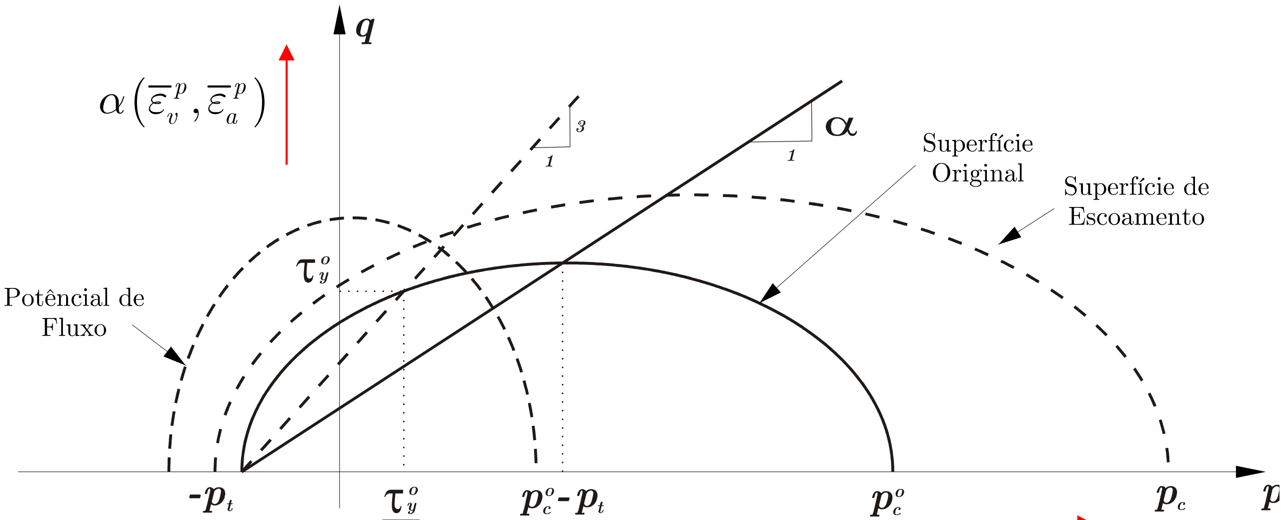


$$E(\rho^*) = c(\rho^*)^{\gamma} E_M$$

Yield Function and the Plastic Flow Potential

$$\mathcal{F}(q,p) = \sqrt{q^2 + \alpha^2\left(p - \left[\frac{p_c - p_t^o}{2}\right]\right)^2} - \alpha\left[\frac{p_c + p_t^o}{2}\right] \leq 0$$

$$\mathcal{G}(q,p) = \sqrt{q^2 + \beta^2_{(\nu_p)} p^2}$$



$\alpha\left(\bar{\varepsilon}_v^p, \bar{\varepsilon}_a^p\right)$

τ_y^o

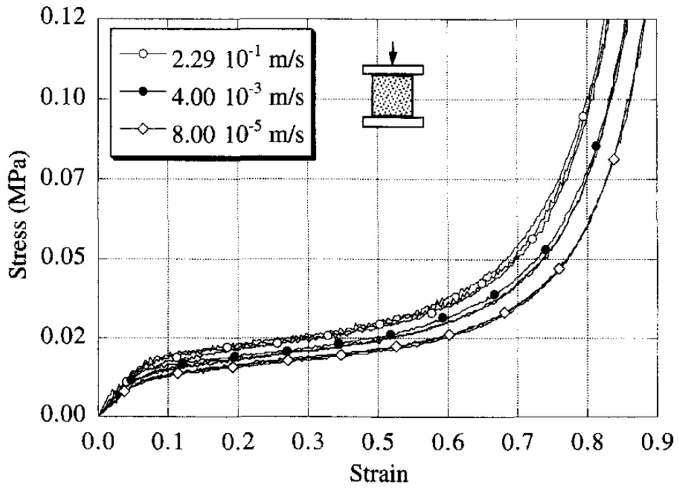
$-p_t$ $\quad \frac{\tau_y^o}{3}$ $\quad \frac{p_c^o - p_t}{2}$ $\quad p_c^o$ $\quad p_c$

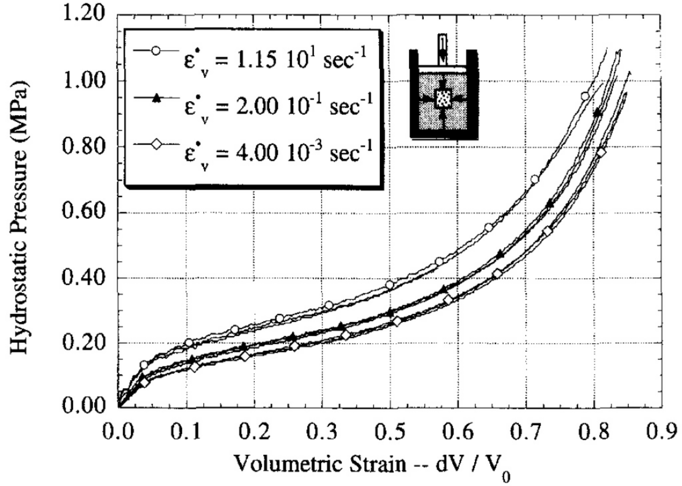
$p_t = \varsigma p_c$

$H_p\left(\bar{\varepsilon}_v^p\right)$

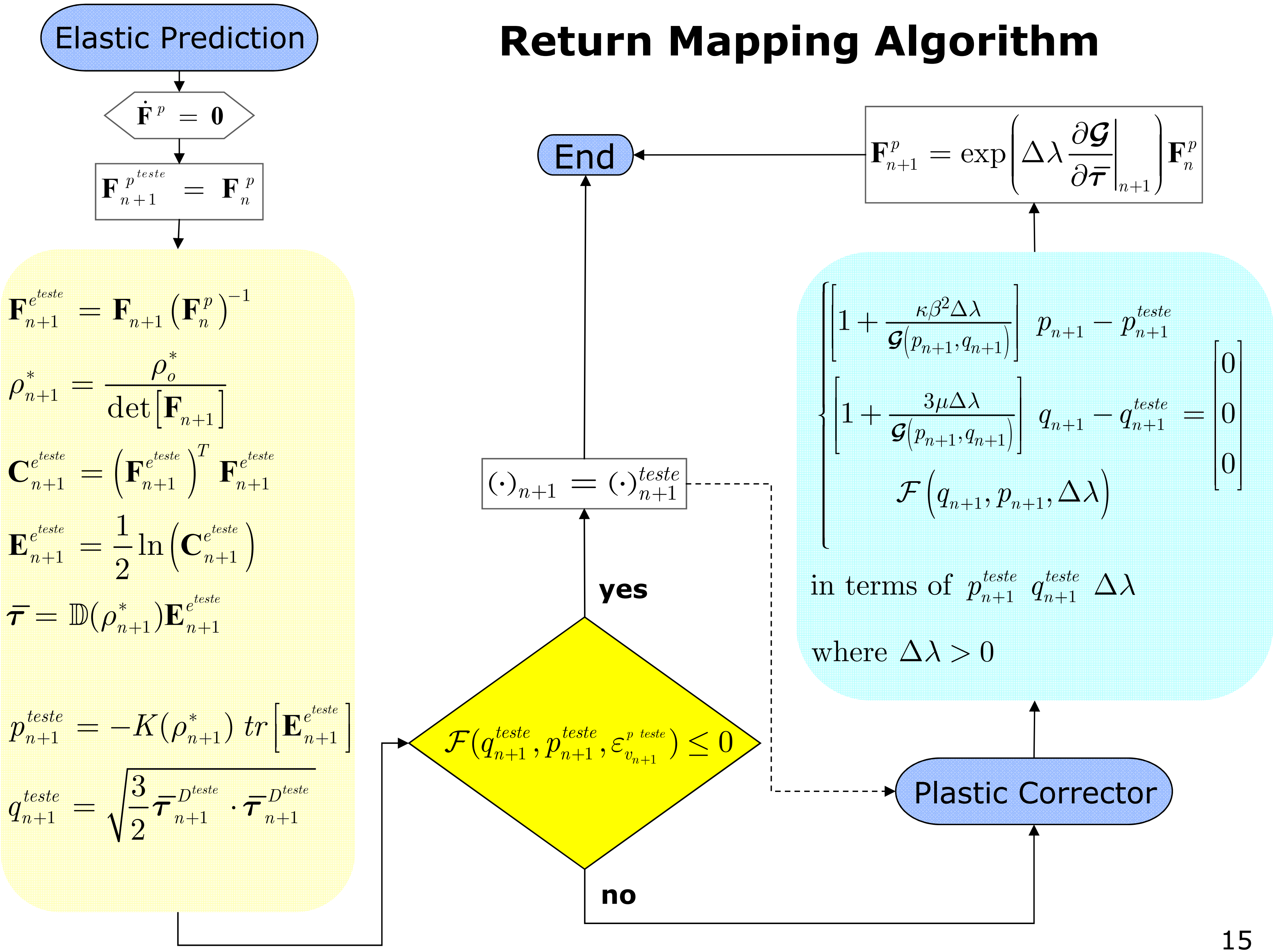
$$\bar{\tau}_y\left(\bar{\varepsilon}_a^p\right) = \bar{\tau}_y^o + H\left(\bar{\varepsilon}_a^p\right) \qquad \bar{\varepsilon}_a^p = -\ln\left(\frac{L^p}{L_o}\right)$$

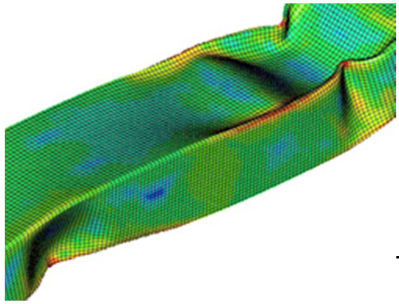
$$p_c = p_c^o + H_p\left(\bar{\varepsilon}_v^p\right) \qquad \bar{\varepsilon}_v^p = -\ln\left(J^p\right)$$





Return Mapping Algorithm



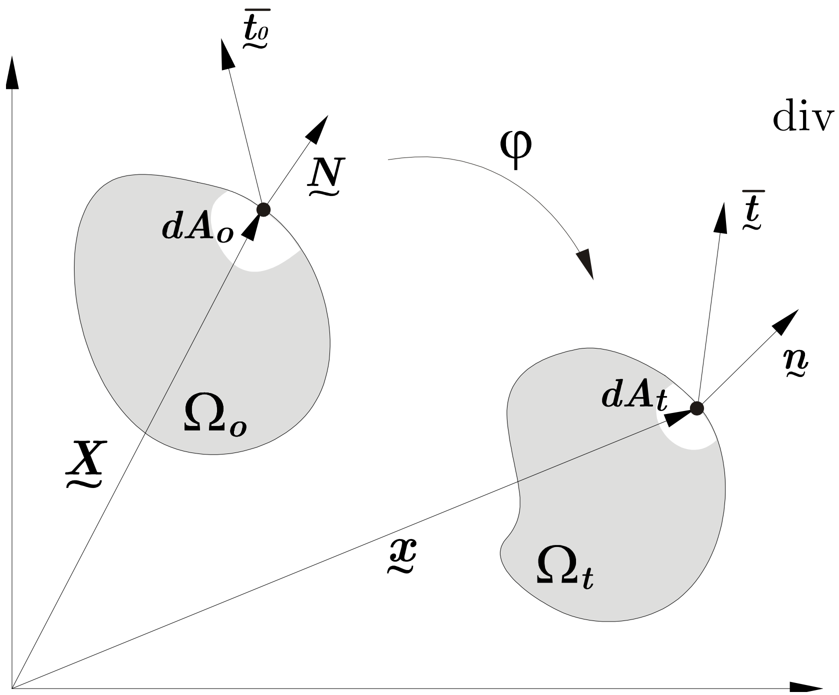


Finite Strain Elastoplastic Formulation

- **The global problem value**

  - **Strong formulation: reference configuration**

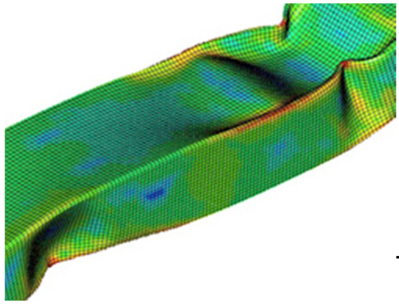
    - For each $t \in [t_o, t_f]$, determine $\vec{u}(\vec{X},t)$ that it is solution of



$$\operatorname{div} \mathbf{P}(\vec{X},t) - \rho_o(\vec{X})\vec{\bar{b}}(\vec{X},t) = 0 \qquad em \ \Omega_o$$

$$\mathbf{P}(\vec{X},t)\,\vec{N}(\vec{X},t) = \vec{\bar{t}}(\vec{X},t) \quad em \ \Gamma_o^t$$

$$\vec{u}(\vec{X},t) = \vec{\bar{u}}(\vec{X}) \qquad em \ \Gamma_o^u$$



Finite Strain Elastoplastic Formulation

- **The global problem value**
  - **Weak formulation: reference configuration**
    - Find $\vec{u}_{n+1} \in \mathcal{K}$ such that

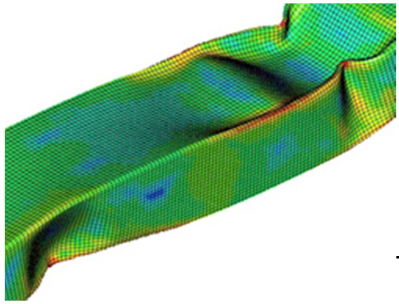
$$F\left(\vec{u}_{n+1};\delta\vec{u}\right)=0 \quad \forall\ \delta\vec{u}\in\mathcal{V}$$

where

$$F\left(\vec{u}_{n+1};\delta\vec{u}\right)=\int_{\Omega_o}\mathbf{P}\left(\vec{u}_{n+1}\right)\cdot\nabla_{\vec{X}}\delta\vec{u}\ d\Omega_o-\int_{\Omega_o}\rho_o\bar{\vec{b}}_{n+1}\cdot\delta\vec{u}\ d\Omega_o-\int_{\Gamma_o^t}\bar{\vec{t}}_{n+1}\cdot\delta\vec{u}\ dA_o$$

$$\mathcal{K}=\left\{\vec{u}\middle|\,u_i\in W_p^1\left(\Omega\right),\ \vec{u}=\bar{\vec{u}}\ em\ \Gamma_o^u\right\}$$

$$\mathcal{V}=\left\{\delta\vec{u}\middle|\,\delta u_i\in W_p^1\left(\Omega\right),\ \delta\vec{u}=0\ em\ \Gamma_o^u\right\}$$



Finite Strain Elastoplastic Formulation

- **Local Linearization (*Newton's* Method)**

  - Considers $F(\cdot,\cdot)$ as being enough regular

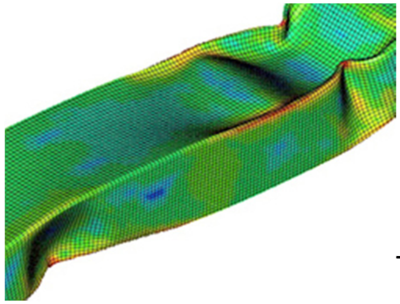
$$F\left(\vec{u}_{n+1}^{k+1};\delta\vec{u}\right) = F\left(\vec{u}_{n+1}^{k} + \Delta\vec{u}_{n+1}^{k};\delta\vec{u}\right) = 0 \qquad \forall\delta\vec{u}\in\mathcal{V}$$

and expanding by *Taylor* in terms of $\vec{u}_{n+1}^{k}$, we obtain

$$F\left(\vec{u}_{n+1}^{k} + \Delta\vec{u}_{n+1}^{k};\delta\vec{u}\right) \simeq F\left(\vec{u}_{n+1}^{k};\delta\vec{u}\right) + DF\left(\vec{u}_{n+1}^{k};\delta\vec{u}\right)\left[\Delta\vec{u}_{n+1}^{k}\right]$$

$$DF\left(\vec{u}_{n+1}^{k};\delta\vec{u}\right)\left[\Delta\vec{u}_{n+1}^{k}\right] = \int_{\Omega_o} \mathbb{A}\left(\vec{u}_{n+1}^{k}\right)\cdot\nabla_{\vec{X}}\left(\Delta\vec{u}_{n+1}^{k}\right)\cdot\nabla_{\vec{X}}\delta\vec{u}\; d\Omega_o$$

$$\left[\mathbb{A}\left(\vec{u}_{n+1}^{k}\right)\right]_{ijkl} = \left.\frac{\partial P_{ij}}{\partial F_{kl}}\right|_{\vec{u}_{n+1}^{k}} = \frac{\partial \tau_{ip}}{\partial F_{kl}} F_{jp}^{-1} - \tau_{ip} F_{jk}^{-1} F_{lp}^{-1}$$



Finite Strain Elastoplastic Formulation

- **Global Equilibrium**
  - Find $\vec{u}^h_{n+1}$, that satisfies the residual equilibrium equation

$$\vec{r}\left(\vec{u}^h_{n+1}\right) = \vec{f}^{\,\mathrm{int}}\left(\vec{u}^h_{n+1}\right) - \vec{f}^{\,ext}_{n+1} = 0$$

$$\vec{f}^{\,int}\left(\vec{u}^h_{n+1}\right) = \int_{\Omega_o} \left(\mathbf{G}^g\right)^T \vec{P}\left(\vec{u}^h_{n+1}\right)\, d\Omega_o \qquad \vec{f}^{\,ext}_{n+1} = \int_{\Omega_o} \rho_o \left(\mathbf{\Phi}^g\right)^T \vec{\bar{b}}_o \, d\Omega_o + \int_{\Gamma^t_o} \left(\mathbf{\Phi}^g\right)^T \vec{\bar{t}}_{o_{n+1}} \, dA_o$$

- **Global Linearization (*Newton's* Method)**

$$\vec{r}\left(\vec{u}^{h^{k+1}}_{n+1}\right) = \vec{r}\left(\vec{u}^{h^k}_{n+1} + \Delta\vec{u}^{h^k}_{n+1}\right) = \vec{0}$$

$$\vec{r}\left(\vec{u}^{h^k}_{n+1} + \Delta\vec{u}^{h^k}_{n+1}\right) \simeq \vec{r}\left(\vec{u}^{h^k}_{n+1}\right) + D\vec{r}\left(\vec{u}^{h^k}_{n+1}\right)\left[\Delta\vec{u}^{h^k}_{n+1}\right]$$

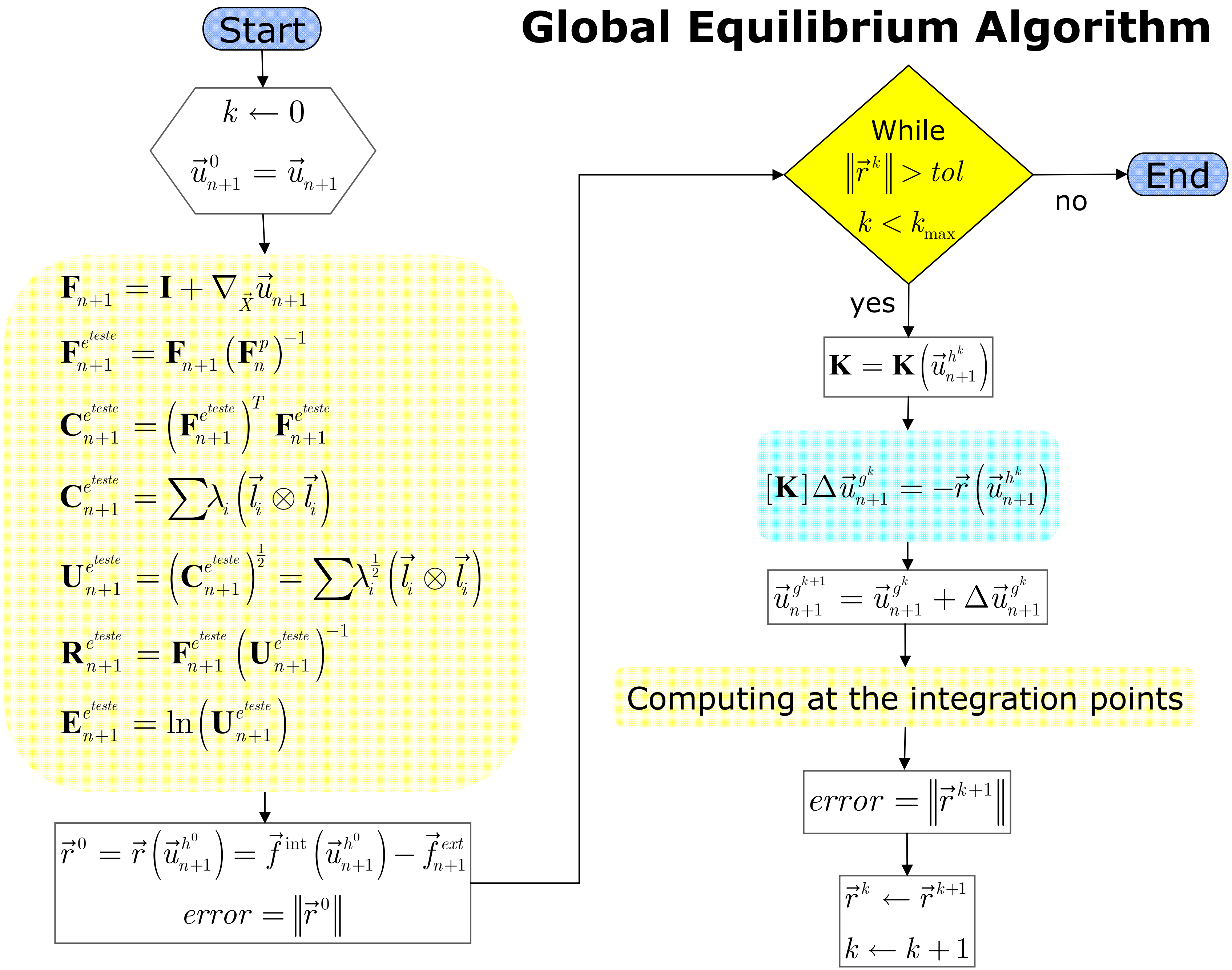
$$D\vec{r}\left(\vec{u}^k_{n+1}\right)\left[\Delta\vec{u}^k_{n+1}\right] = \mathbf{K}\Delta\vec{u}^{g^k}_{n+1}$$

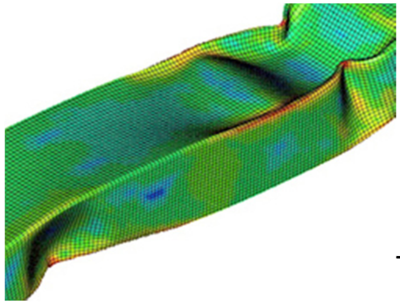
$$\mathbf{K} = \int_{\Omega_o} \left(\mathbf{G}^g\right)^T \mathbf{A}\mathbf{G}^g \, d\Omega_o$$

$$[\mathbf{K}]\Delta\vec{u}^{g^k}_{n+1} = -\vec{r}\left(\vec{u}^{h^k}_{n+1}\right)$$

$$\text{for } \Delta\vec{u}^{g^k}_{n+1}$$

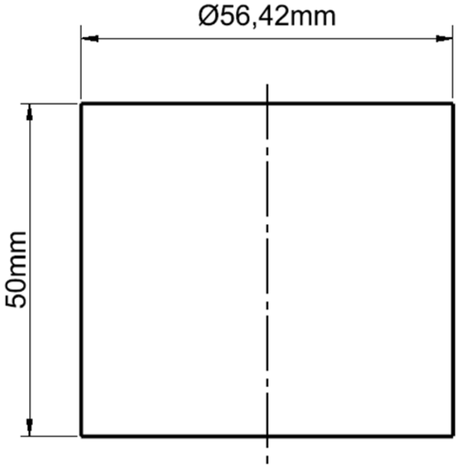
Global Equilibrium Algorithm

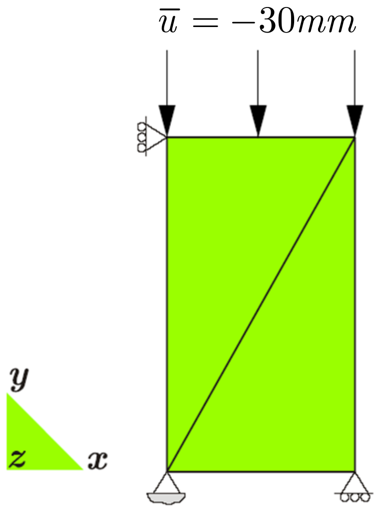


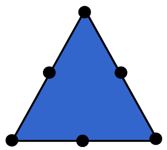


Finite Strain Elastoplastic Formulation

- **Examples**
  - Axisymmetric Uniaxial Compression







TRI6

$$\tau_y^0 = 0{,}082034\ Mpa$$

$$p_c^0 = 0{,}040470\ MPa$$

$$\varsigma = 0{,}10$$

$$E_m = 928{,}092\ MPa$$

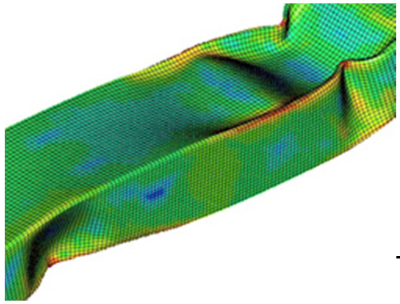
$$\rho_0^* = 0{,}049\ Kg\,/\,m^3$$

$$\nu_p = 0{,}00$$

$$\nu = 0{,}25$$

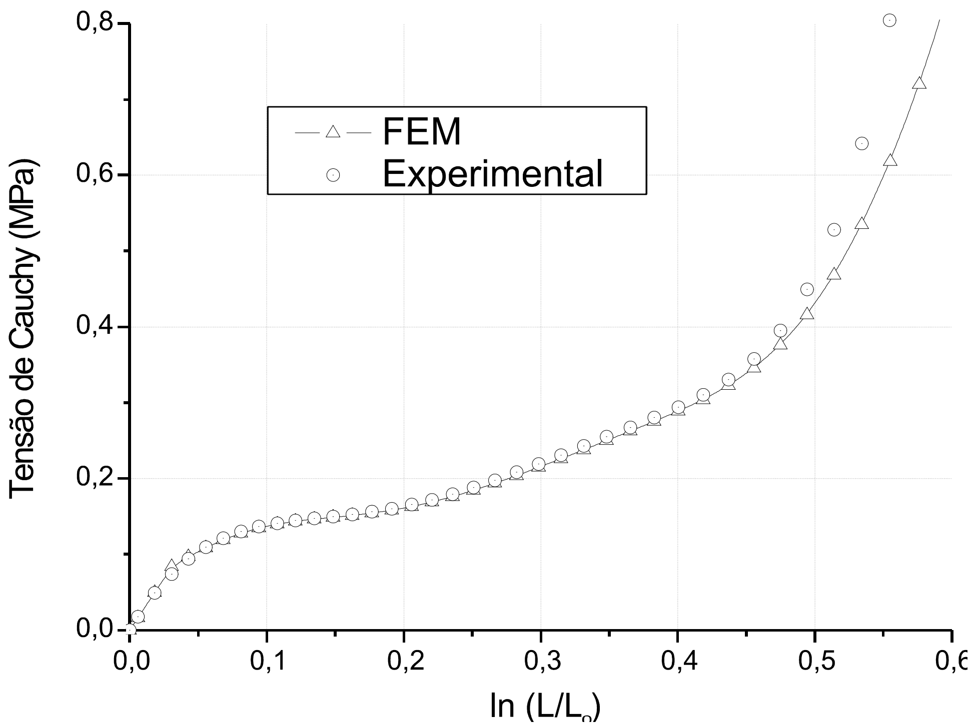
$$\gamma = 1{,}54$$

$$c = 0{,}30$$

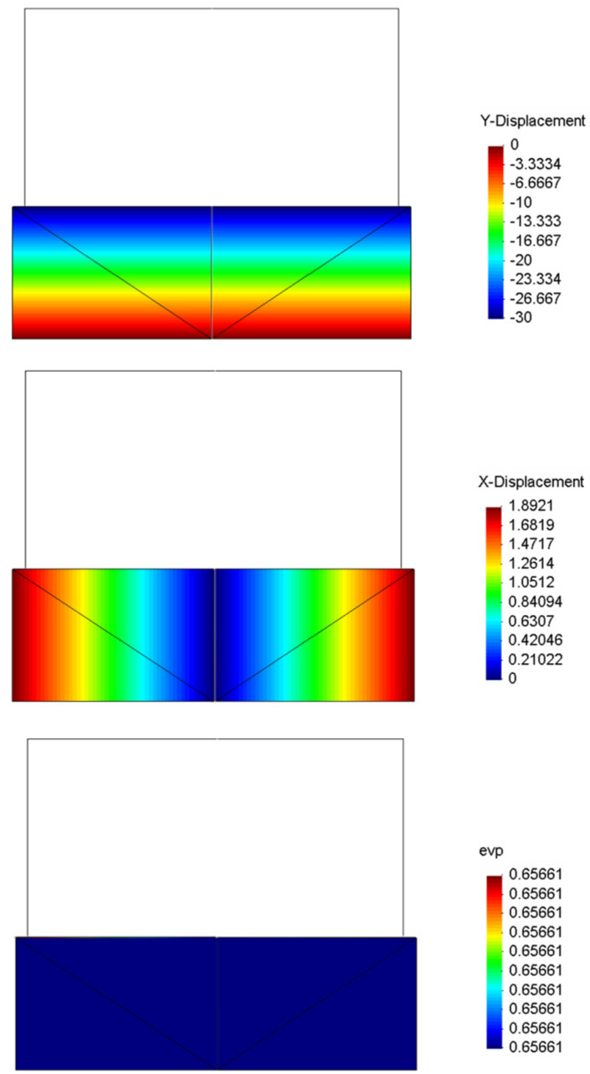


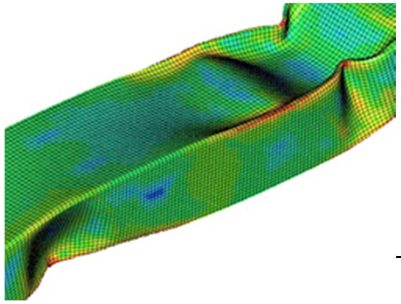
Finite Strain Elastoplastic Formulation

- Uniaxial Compression (Axisymmetric)



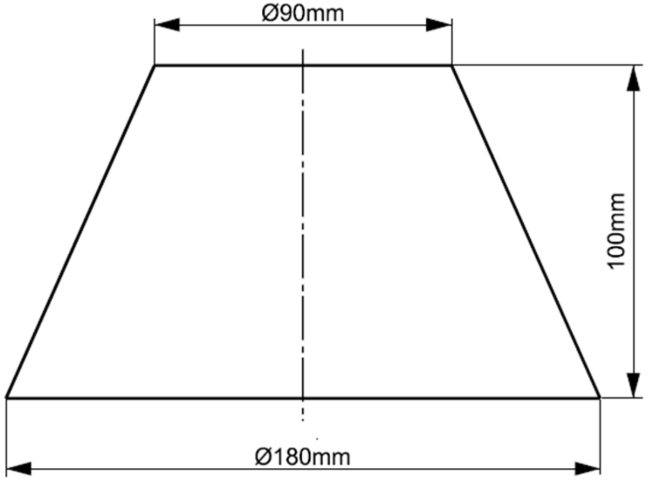
Experimental data ZHANG, J. *at al* (1998)

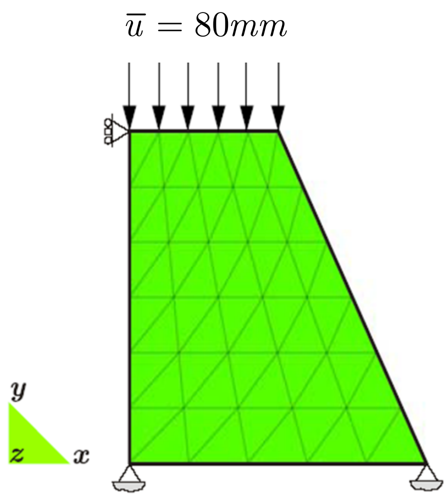




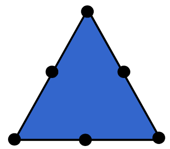
Finite Strain Elastoplastic Formulation

- Multiaxial Compression – Axisymmetric Cone Slice

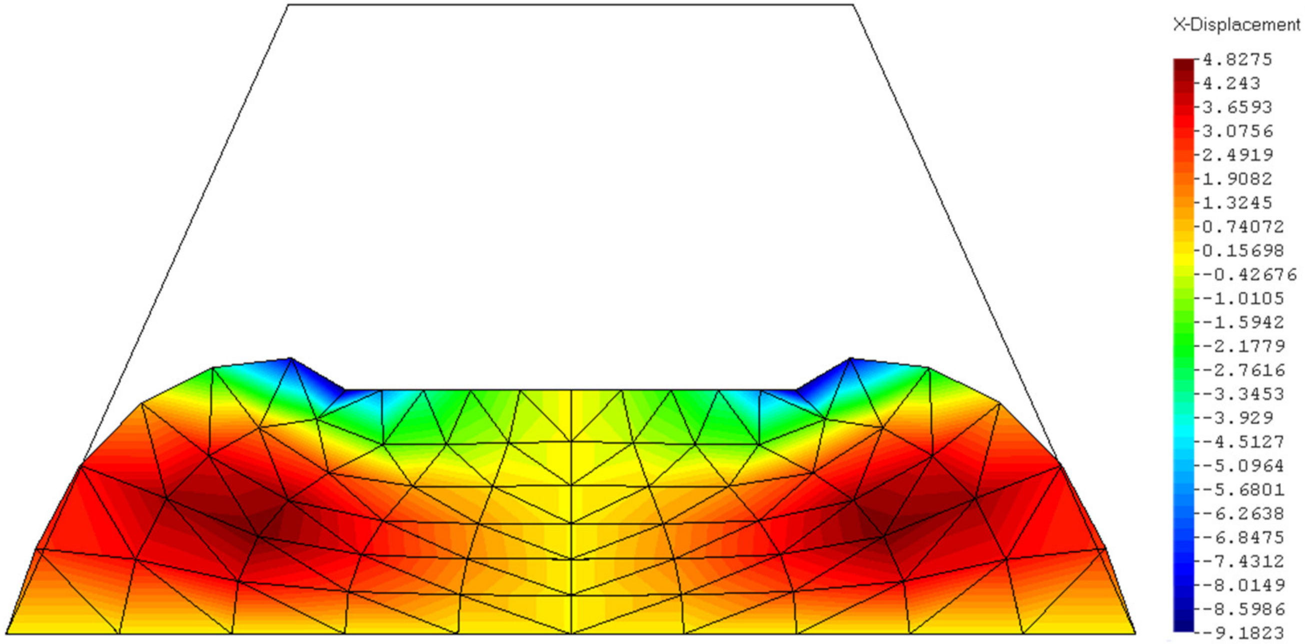




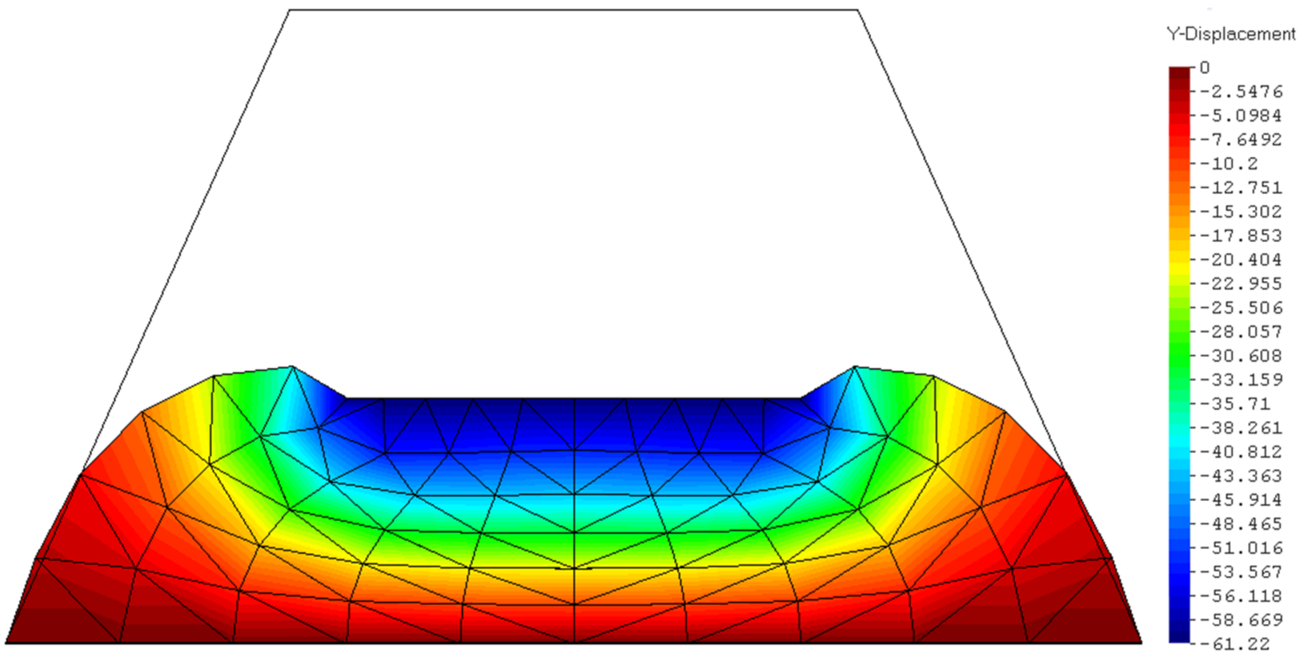
$$\tau_y^o = 0,082\ Mpa$$
$$p_c^o = 0,040\ MPa$$
$$\varsigma = 0,01$$
$$E_m = 928,092\ MPa$$
$$\rho_o^* = 0,049\ Kg / m^3$$
$$\nu_p = 0,00$$
$$\nu = 0,25$$
$$\gamma = 1,54$$
$$c = 0,30$$

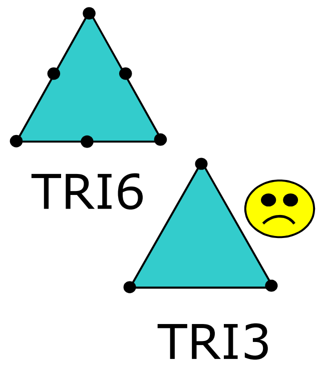


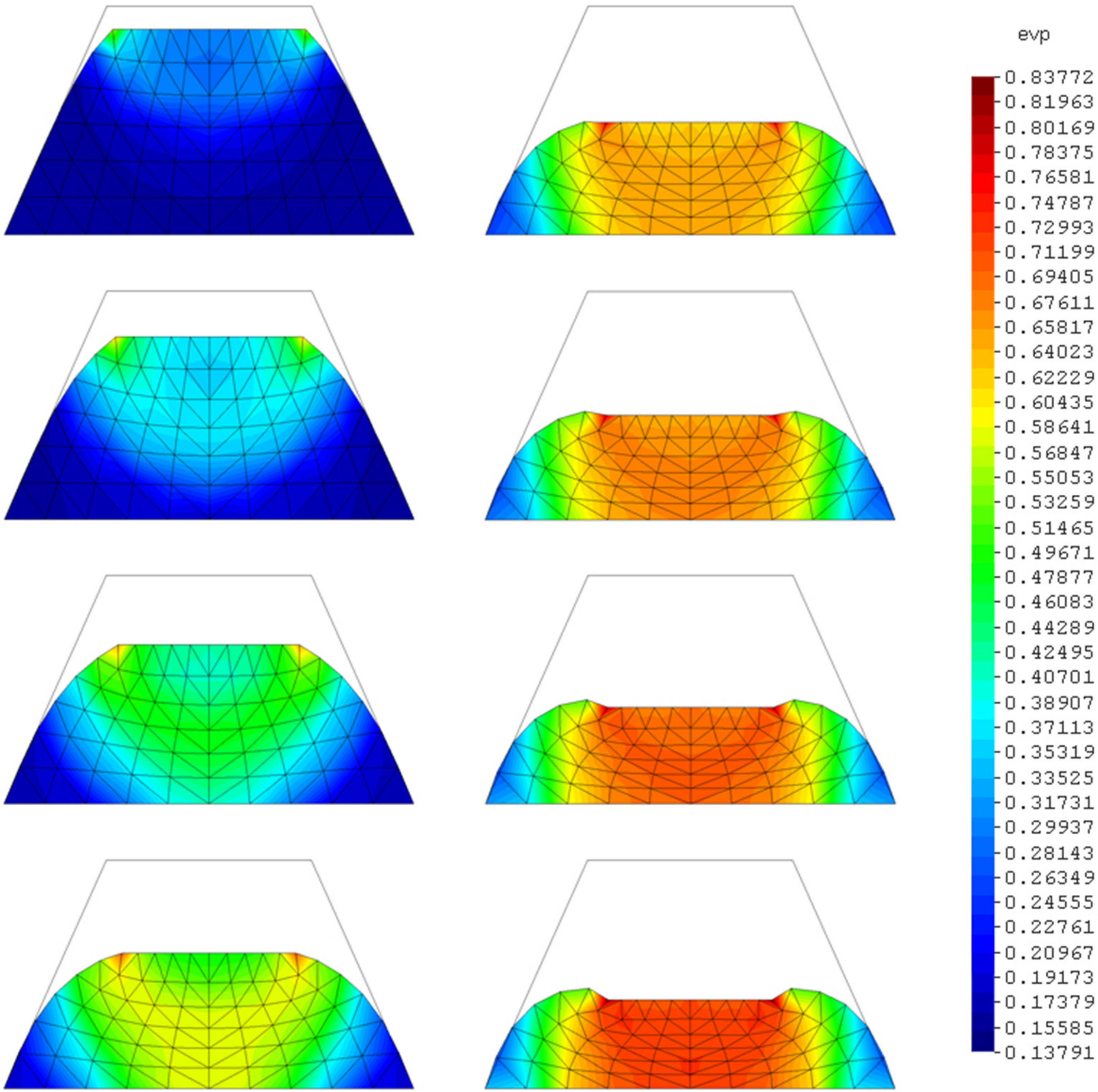
TRI6

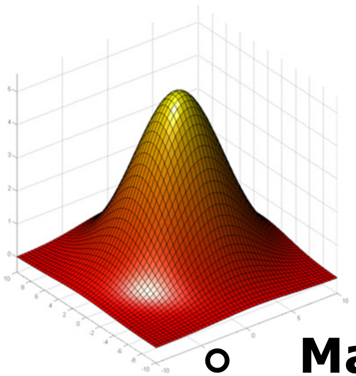


Note: Calculated with TRI6 visualized as TRI3 element!






evp
0.83772
0.81963
0.80169
0.78375
0.76581
0.74787
0.72993
0.71199
0.69405
0.67611
0.65817
0.64023
0.62229
0.60435
0.58641
0.56847
0.55053
0.53259
0.51465
0.49671
0.47877
0.46083
0.44289
0.42495
0.40701
0.38907
0.37113
0.35319
0.33525
0.31731
0.29937
0.28143
0.26349
0.24555
0.22761
0.20967
0.19173
0.17379
0.15585
0.13791

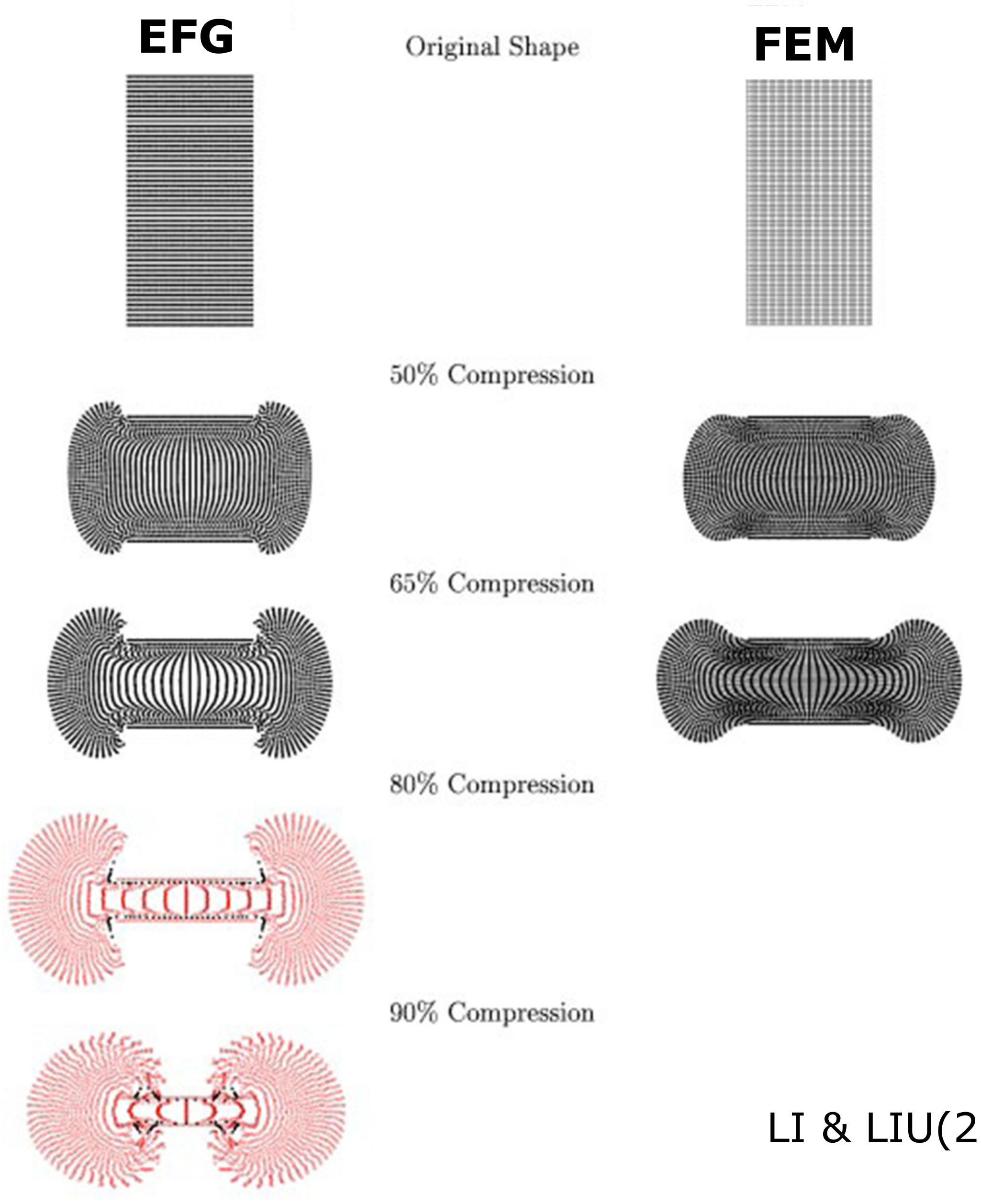


Element-Free Galerkin Method (EFG)

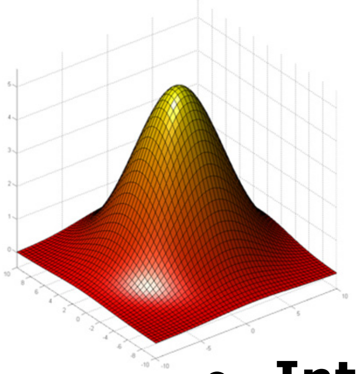
- **Main characteristics** BELYTSCHKO *et al.* (1994)
  - The discretization of the differential equations system uses the weak Galerkin form;
  - Do not required a finite element mesh;
  - The discretization is based on a set of nodes (discrete data);
  - The connection in terms of nodes interactions may be change constantly, and modeling fracture (ability to simulate crack growth), free surfaces, large deformations, etc. is considerably simplified;
  - Accuracy can be controlled more easily, since in areas where more refinement is needed, nodes can be added easily;
  - Meshfree discretization can provide accurate representation of geometric object;

  - A special strategy to impose the essential boundary conditions.

Comparison of the deformations at different time stages for a block of hyperelastic material under compression by using:



LI & LIU(2002)



Element-Free Galerkin Method (EFG)

- **Introduction**

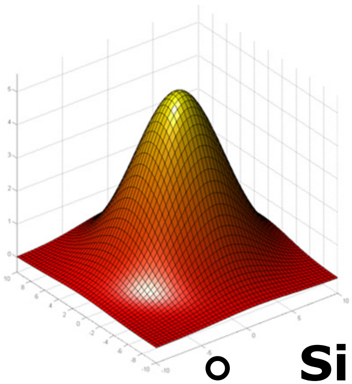
- Enables the construction of an approximate function

$$u^h(\boldsymbol{X})$$

that fits a discrete set of data

$$u_I = \{u_I,\ \mathrm{I{=}1,\ldots,n}\}$$

- *Moving Least Square Approximation* (MLSA). LANCASTER & SALKAUSKAS (1981);



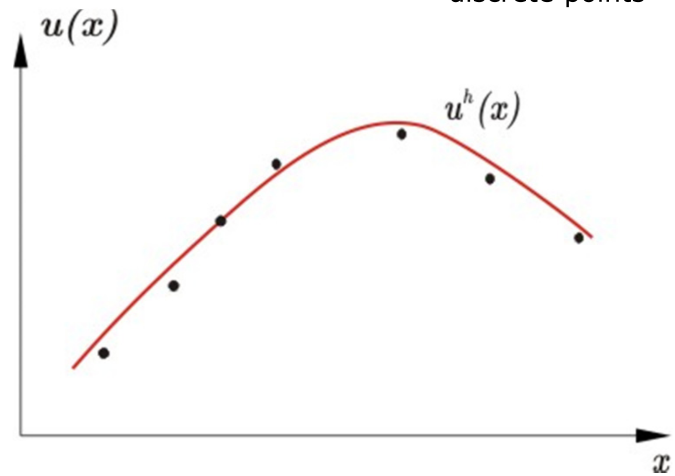
Element-Free Galerkin Method (EFG)

- **Simple Least Squares Method**
  - Finding the best-fitting curve to a given set of discrete points (or nodes)

$$u_I = \{u_I,\ \mathrm{I=1,\ldots,n}\} \Longrightarrow u^h(\vec{x}) = \sum_{j=1}^{m} p_j(\vec{x})\, a_j = \left\langle \vec{p}(\vec{x}), \vec{a} \right\rangle$$

discrete points

approximation function



$$J(\vec{a}^*) = \sum_{I=1}^{n} \left\{ u^h(x_I) - u_I \right\}^2 = \sum_{I=1}^{n} \left\{ \left\langle \vec{p}, \vec{a}^* \right\rangle - u_I \right\}^2$$

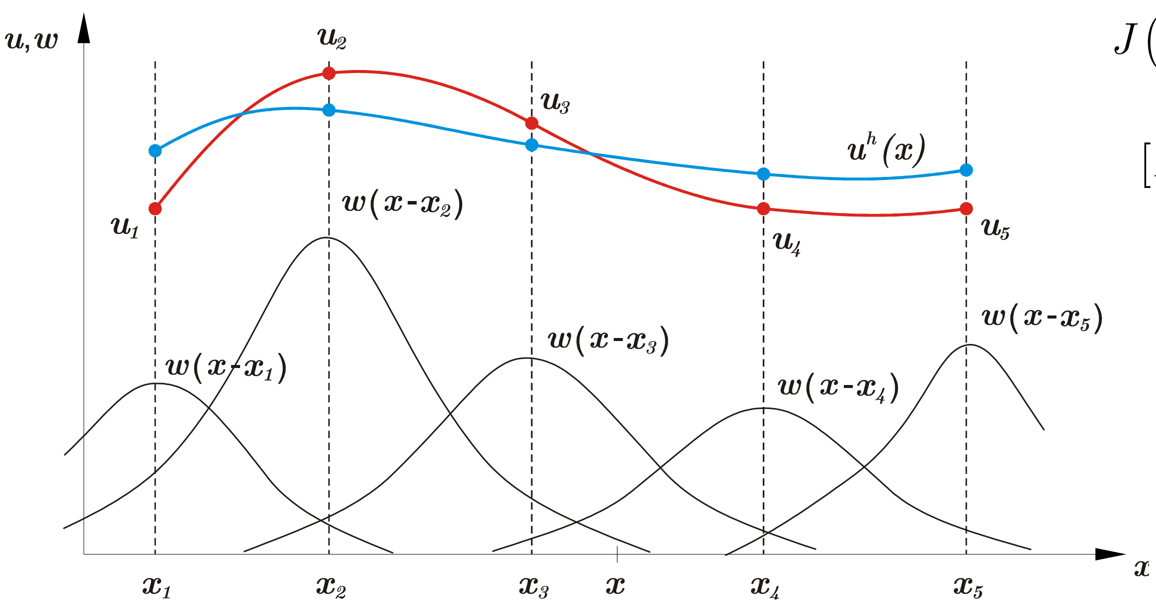
$$\vec{a} = \arg\min \left\{ J(\vec{a}^*) \right\} \quad \forall \vec{a}^* \in \Re^m$$

$$[\mathbf{A}]\vec{a} = \sum_{I=1}^{n} u_I \vec{b}_I$$

where $[\mathbf{A}] = \sum_{I=1}^{n} \left[\vec{p}(x_I) \otimes \vec{p}(x_I) \right]$ and $\vec{b}_I = \vec{p}(x_I)$.

- **ALL** values u_I, $I=1,..n$ exerts influence to determinate **ALL** the a_j coefficients, for the n discrete points.

The Moving Least Square Approximation (MLSA)



$$J(\vec{a}) = \sum_{I=1}^{n} w(\vec{x} - \vec{x}_I)\left[\left\langle \vec{p}(\vec{x}_I), \vec{a}(\vec{x}) \right\rangle - u_I\right]^2$$

$$[\mathbf{A}(x)] = \sum_{I=1}^{n} w(x - x_I)\left[\vec{p}(x_I) \otimes \vec{p}(x_I)\right]$$

$$\vec{b}_I = w(x - x_I)\,\vec{p}(x_I)$$

$$[\mathbf{A}(x)]\,\vec{a}(x) = \sum_{I=1}^{n} u_I \vec{b}_I$$

$$\vec{a}(x) = [\mathbf{A}(x)]^{-1}\left\{\sum_{I=1}^{n} u_I \vec{b}_I\right\}$$

$$u^h(x) = \left\langle \vec{p}(x), \vec{a}(x) \right\rangle = \sum_{I=1}^{n} u_I \left\langle \vec{p}(x), [\mathbf{A}(x)]^{-1}\vec{b}_I \right\rangle = \sum_{I=1}^{n} u_I \Phi_I(x)$$

$$\boxed{\Phi_I(x) = \left\langle \vec{p}(x), [\mathbf{A}(x)]^{-1}\vec{b}_I \right\rangle}$$

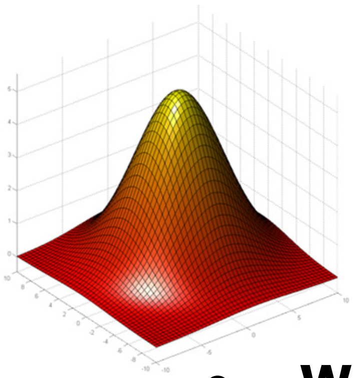
moment matrix

global shape functions

intrinsic base functions

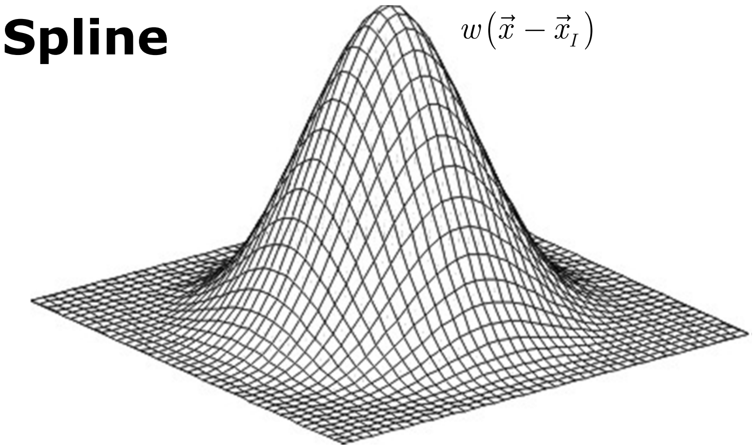
$$\vec{p}^{\,T}(\vec{x}) = \left[1, x, y, x^2, xy, y^2, \ldots, x^k, \ldots xy^{k-1}, y^k\right]$$

whit consistency order Co^k



Element-Free Galerkin Method (EFG)

- **Weight function – Quartic Spline**



$$w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & \text{para } r \le 1; \\ 0 & \text{para } r > 1; \end{cases}$$

- **Influence Domain definition**

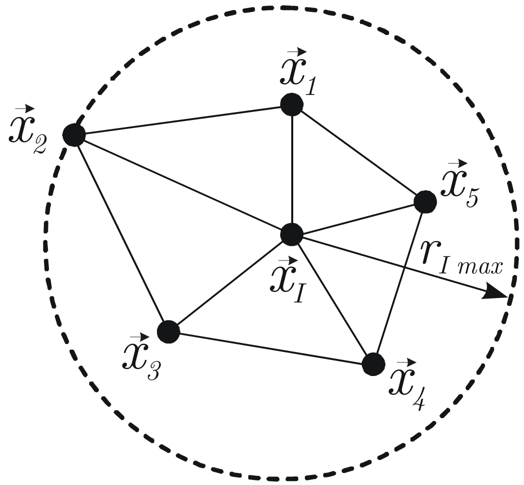
$$r = \frac{r_I}{\bar{r}_I} = \frac{\|\vec{x}-\vec{x}_I\|}{s \cdot r_{I_{\max}}}$$

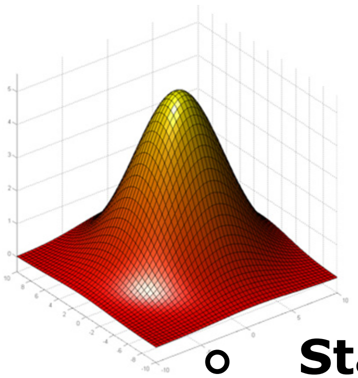
Parameterized radius (r)

Influence factor (s)

$$r_{I_{\max}} = \max_i \|\vec{x}_i - \vec{x}_I\|, \quad i \in \vec{L}_I$$

Associated list ($\vec{L}_I$)

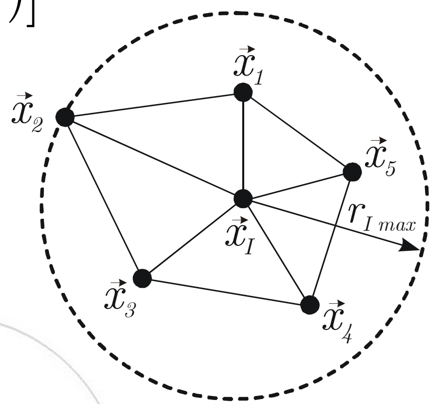




Element-Free Galerkin Method (EFG)

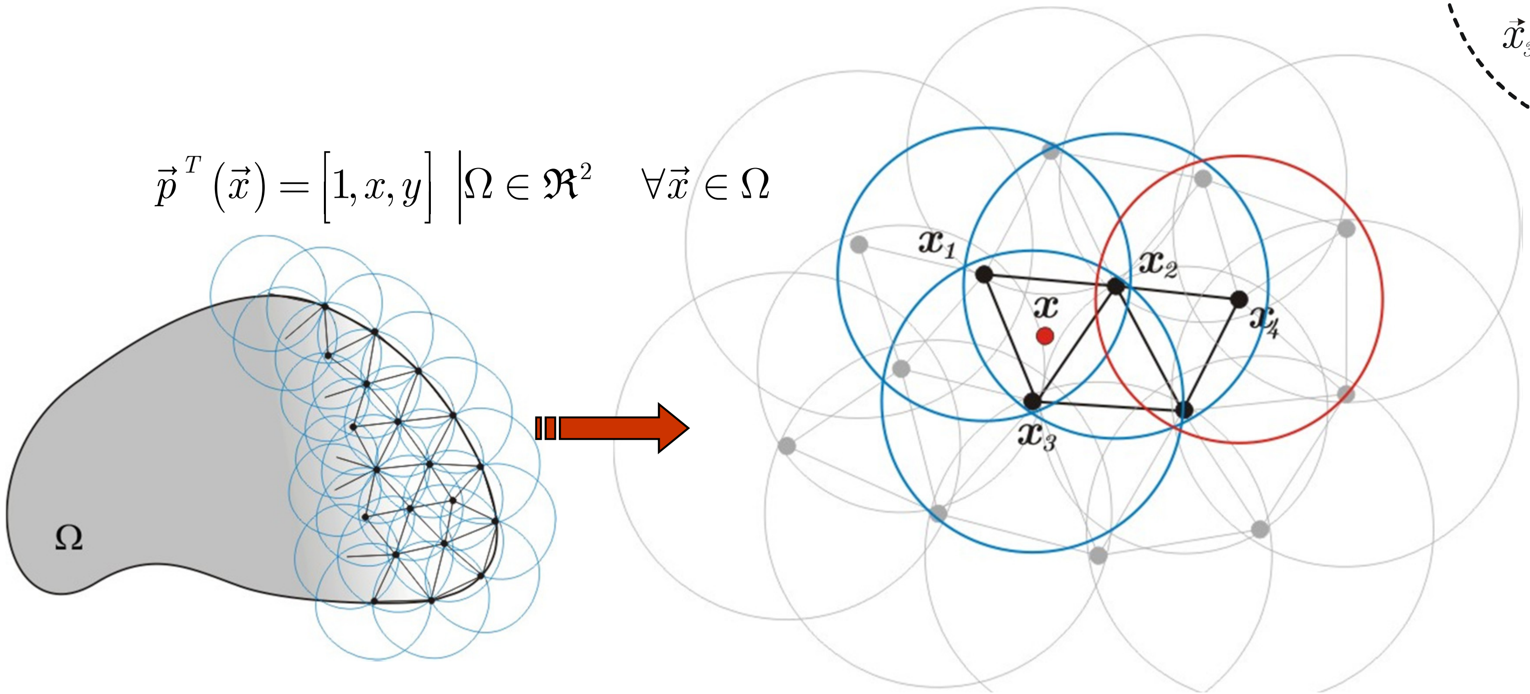
- **Stability Conditions** LIU *et al.* (1996)

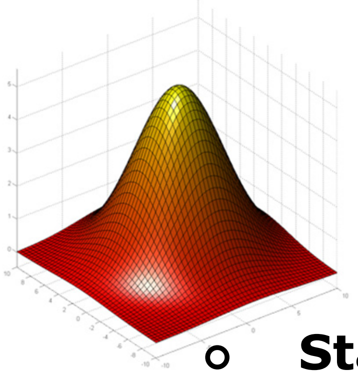
  - The choice of the size of the support to assure $\left[\mathbf{A}\left(\vec{x}\right)\right]^{-1}$
  - The consistency order of the intrinsic base



$$\text{card}\left\{\vec{x}_i \middle| \Phi_i\left(\vec{x}\right) \neq 0\right\} \geq \dim\left[\mathbf{A}\left(\vec{x}\right)\right]$$

$$\vec{p}^{\,T}\left(\vec{x}\right) = \left[1, x, y\right] \quad \left|\Omega \in \Re^2 \quad \forall \vec{x} \in \Omega\right.$$

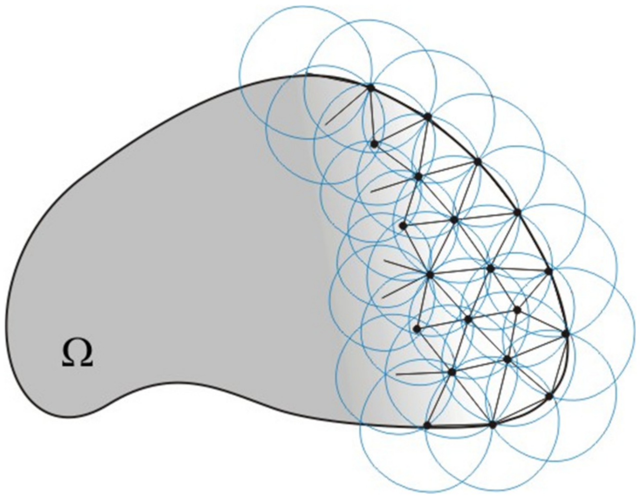


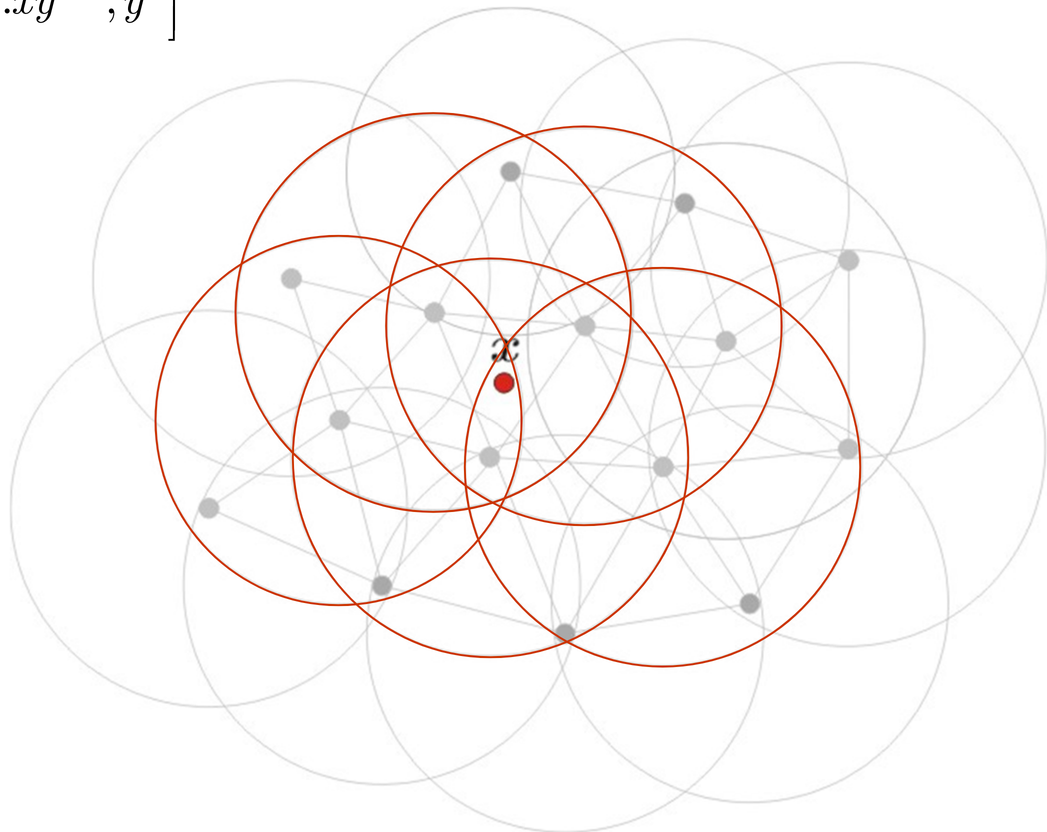


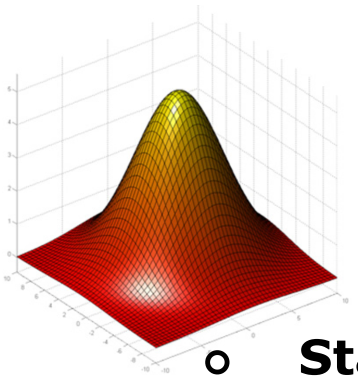
Element-Free Galerkin Method (EFG)

- **Stability Conditions** LIU *et al.* (1996)

$$\vec{p}^{\,T}(\vec{x}) = \left[1, x, y, x^2, xy, y^2, \ldots, x^k, \ldots xy^{k-1}, y^k\right]$$

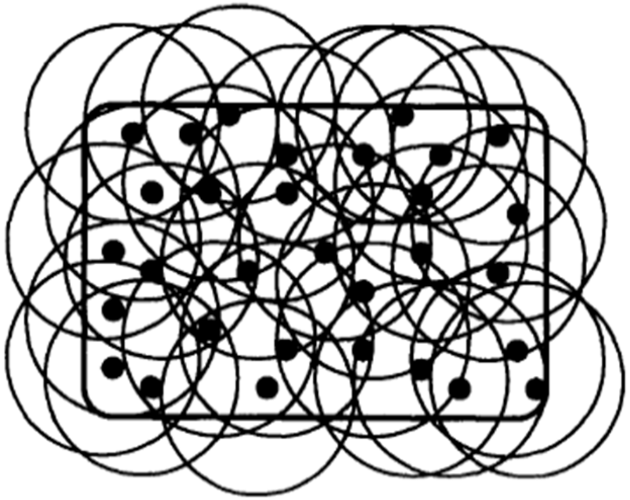


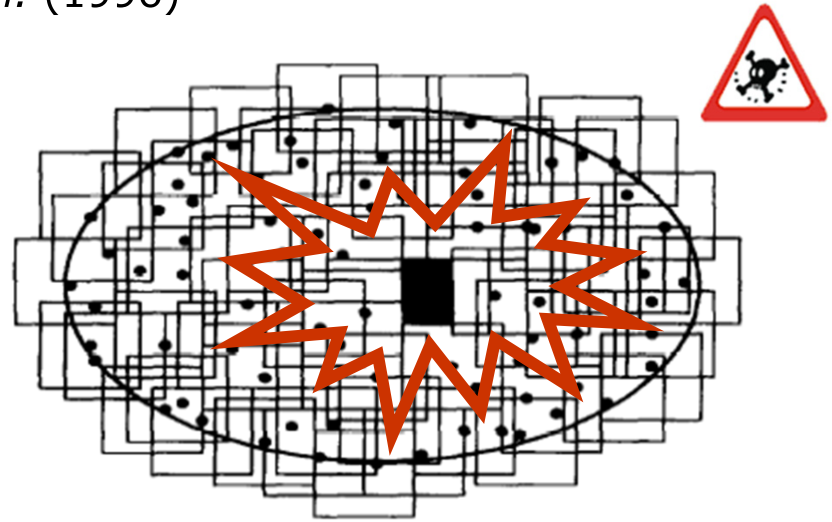


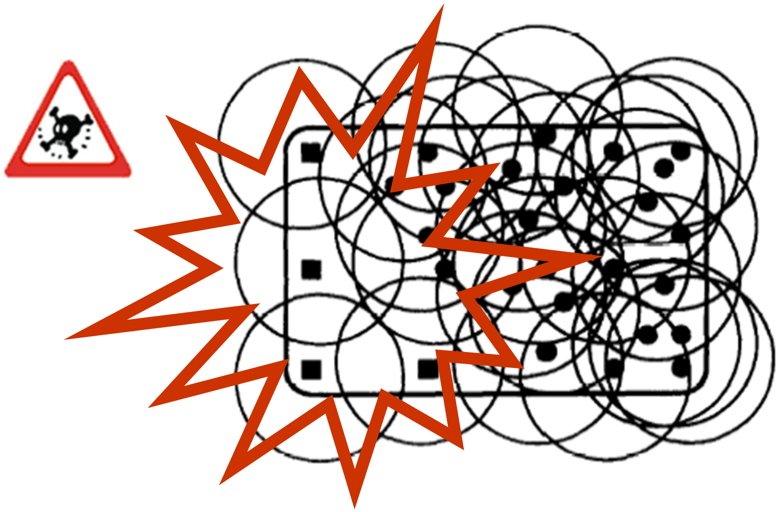


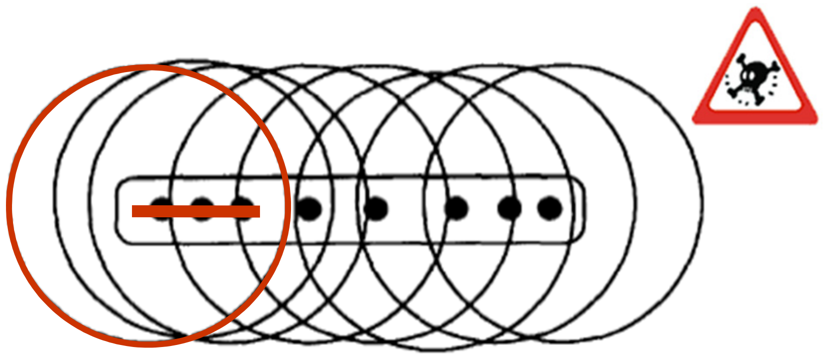
Element-Free Galerkin Method (EFG)

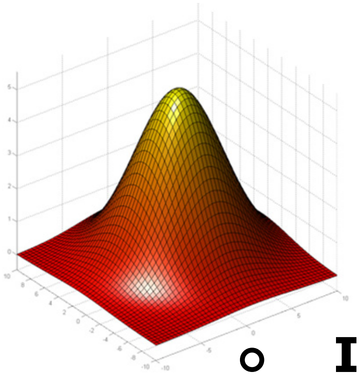
- **Stability Conditions** LIU *et al.* (1996)











Element-Free Galerkin Method (EFG)

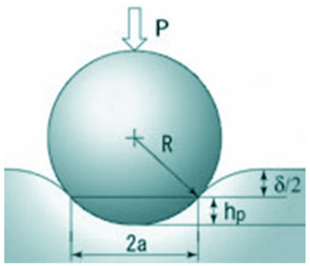
- **Imposition of the Essential Boundary Conditions using The Augmented Lagrange Method**

$$\Phi_I\left(\vec{x}_J\right) \neq \delta_{IJ}$$

$$\vec{Q}^{u}_{n+1} = -\left[\vec{\lambda}^{h^k}_{u_{n+1}} + \frac{1}{\varepsilon_u}\left(\vec{u}^{h^{k(i)}}_{n+1} - \bar{\vec{u}}_{n+1}\right)\right] \quad em \ \Gamma^u_o$$

$$\vec{u}^{h^{k(i)}}_{n+1} = \mathbf{\Phi}^g \vec{u}^{g^{k(i)}}_{n+1} \qquad \delta\vec{u}^h = \mathbf{\Phi}^g \delta\vec{u}^g \qquad \vec{\lambda}^{h^k}_{u_{n+1}} = \mathbf{N}^g \vec{\lambda}^{g^k}_{u_{n+1}}$$

$$F^u\left(\vec{u}^{h^{k(i)}}_{n+1}, \delta\vec{u}^h\right) = -\int_{\Gamma^u_o} \vec{Q}^{u^{k(i)}}_{n+1} \cdot \delta\vec{u}^h \ d\Gamma^u_o = \vec{f}^{\lambda_u{}^k}_{n+1} \cdot \delta\vec{u}^g + \vec{f}^{u^{k(i)}}_{n+1} \cdot \delta\vec{u}^g$$



Unilateral Contact with Friction

- **New Problem Definition**
  - Find $\vec{u}^{k}_{n+1} \in \mathcal{K}$ that

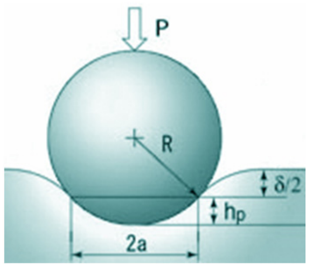
$$\tilde{F}(\vec{u}^{k}_{n+1},\delta\vec{u}) = F(\vec{u}^{k}_{n+1},\delta\vec{u}) + F^{u}(\vec{u}^{k}_{n+1},\delta\vec{u}) + F^{c}(\vec{u}^{k}_{n+1},\delta\vec{u}) = 0 \qquad \forall\ \delta\vec{u} \in \mathcal{V}$$

where

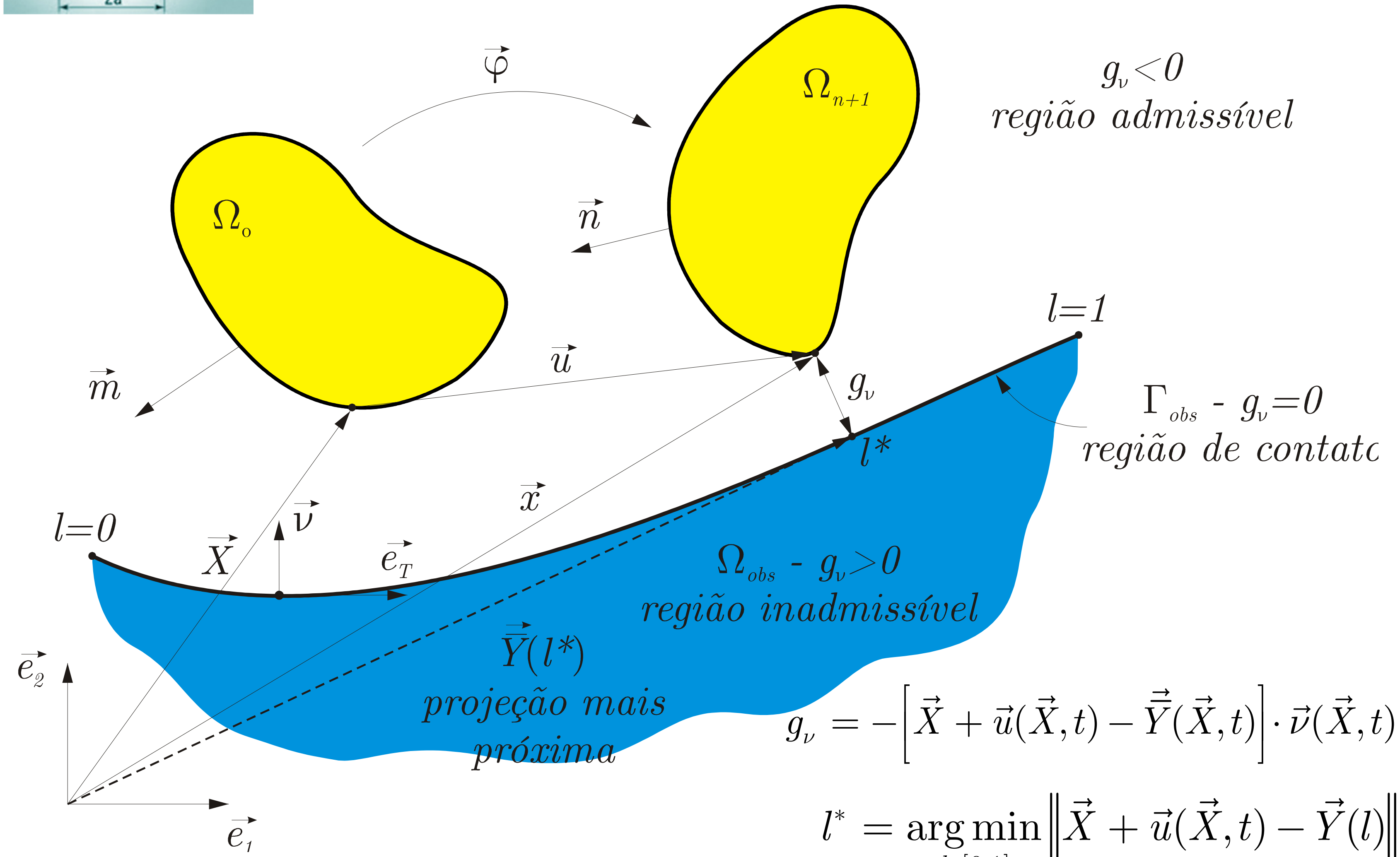
$$F\left(\vec{u}_{n+1};\delta\vec{u}\right) = \int_{\Omega_o} \mathbf{P}\left(\vec{u}_{n+1}\right)\cdot\nabla_{\vec{X}}\delta\vec{u}\ d\Omega_o - \int_{\Omega_o} \rho_o \vec{\bar{b}}_{n+1}\cdot\delta\vec{u}\ d\Omega_o - \int_{\Gamma_o^t} \vec{\bar{t}}_{n+1}\cdot\delta\vec{u}\ dA_o$$

$$F^{u}(\vec{u}^{k}_{n+1},\delta\vec{u}) = -\int_{\Gamma_o^u} \vec{Q}^{u}_{n+1}(\vec{u}^{k}_{n+1},\varepsilon_u,\vec{\lambda}^{k}_{u_{n+1}})\cdot\delta\vec{u}\ dA_o$$

$$F^{c}\left(\vec{u}^{k}_{n+1},\delta\vec{u}\right) = -\int_{\Gamma_o^c} \vec{Q}^{c}_{n+1}(\vec{u}^{k}_{n+1},\varepsilon_\nu,\lambda^{k}_{\nu_{n+1}})\cdot\delta\vec{u}\ dA_o$$

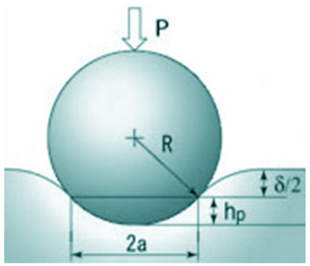


Unilateral Contact with Friction



$$g_\nu = -\left[\vec{X} + \vec{u}(\vec{X},t) - \vec{\bar{Y}}(\vec{X},t)\right] \cdot \vec{\nu}(\vec{X},t)$$

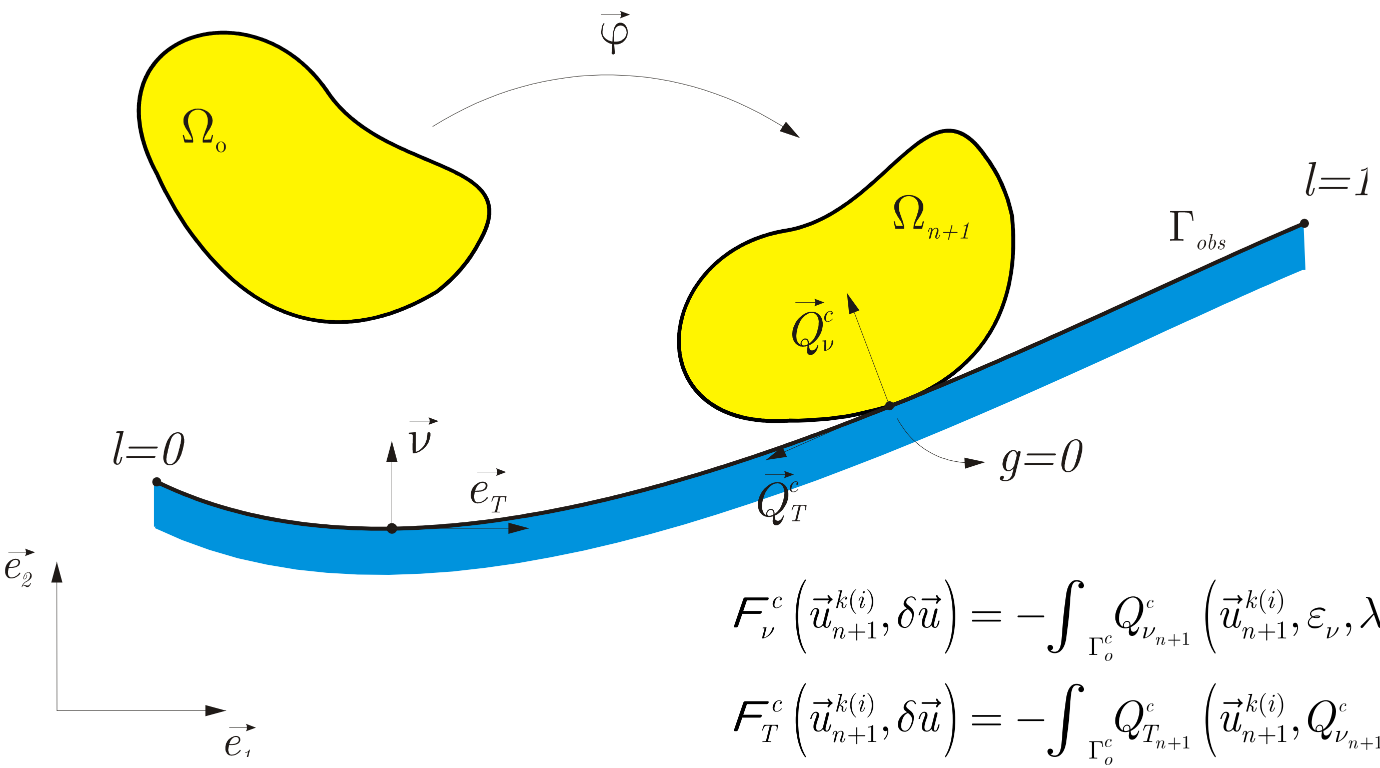
$$l^* = \underset{l \in [0,1]}{\arg\min} \left\| \vec{X} + \vec{u}(\vec{X},t) - \vec{Y}(l) \right\|$$



Unilateral Contact with Friction

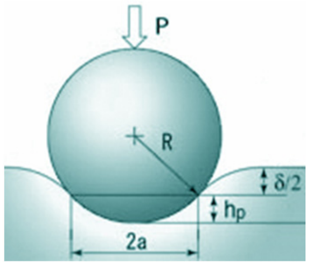
- **Imposition of the Normal Contact and Friction Terms**
  - Normal and Tangential Works

$$F^{c}\left(\vec{u}_{n+1},\delta\vec{u}\right)=F^{c}_{\nu}\left(\vec{u}^{k(i)}_{n+1},\delta\vec{u}\right)+F^{c}_{T}\left(\vec{u}^{k(i)}_{n+1},\delta\vec{u}\right)$$



$$F^{c}_{\nu}\left(\vec{u}^{k(i)}_{n+1},\delta\vec{u}\right)=-\int_{\Gamma^{c}_{o}}Q^{c}_{\nu_{n+1}}\left(\vec{u}^{k(i)}_{n+1},\varepsilon_{\nu},\lambda^{k}_{\nu_{n+1}}\right)\vec{\nu}^{k(i)}_{n+1}\cdot\delta\vec{u}\;d\Gamma^{c}_{o}$$

$$F^{c}_{T}\left(\vec{u}^{k(i)}_{n+1},\delta\vec{u}\right)=-\int_{\Gamma^{c}_{o}}Q^{c}_{T_{n+1}}\left(\vec{u}^{k(i)}_{n+1},Q^{c}_{\nu_{n+1}},\varepsilon_{T}\right)\vec{e}^{k(i)}_{T_{n+1}}\cdot\delta\vec{u}\;d\Gamma^{c}_{o}$$



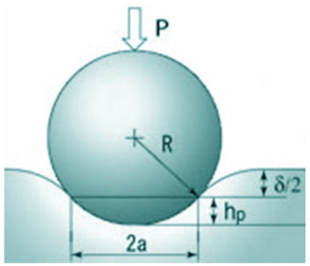
Unilateral Contact with Friction

- **Normal Contact**

  - Augmented Lagrange Method

$$Q^{c}_{\nu_{n+1}}\left(\vec{u}^{k(i)}_{n+1}, \varepsilon_{\nu}, \lambda^{k}_{\nu_{n+1}}\right) = \left\langle \lambda^{k}_{\nu_{n+1}} + \frac{1}{\varepsilon_{\nu}} g\left(\vec{u}^{k(i)}_{n+1}\right) \right\rangle$$

$$F^{c}_{\nu}\left(\vec{u}^{h^{k(i)}}_{n+1}, \delta\vec{u}^{h}\right) = -\int_{\Gamma^{c}_{o}} Q^{c^{k(i)}}_{\nu_{n+1}} \vec{\nu}^{k(i)}_{n+1} \cdot \delta\vec{u}^{h} \; d\Gamma^{c}_{o} = \vec{f}^{c^{k(i)}}_{\nu_{n+1}} \cdot \delta\vec{u}^{g}$$



Unilateral Contact with Friction

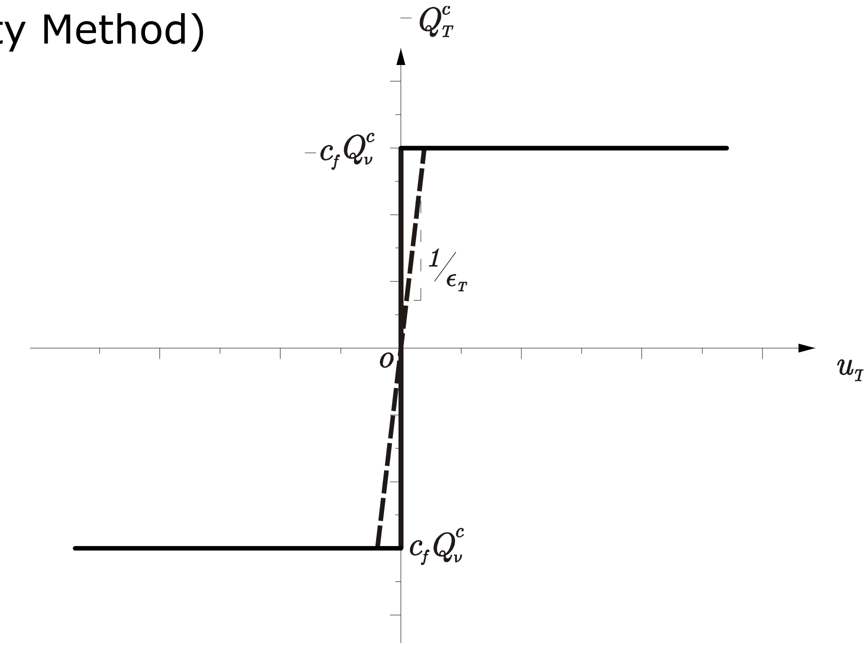
- **Tangential Contact**
  - Regularized Coulomb's Law (Penality Method)

$$\Upsilon\left(Q_{T_{n+1}}^{c\,k(i)}, Q_{\nu_{n+1}}^{c\,k(i)}\right) = \left|Q_{T_{n+1}}^{c\,k(i)}\right| - c_f Q_{\nu_{n+1}}^{c\,k(i)}$$

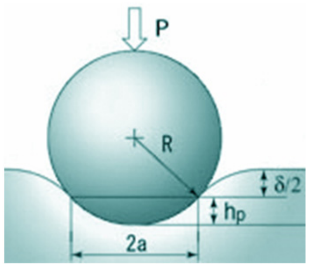
$$\dot{Q}_{T_{n+1}}^{c\,k(i)} = -\frac{1}{\varepsilon_T}\left(\dot{u}_{T_{n+1}}^{k(i)} + \dot{\gamma}\frac{Q_{\nu_{n+1}}^{c\,k(i)}}{\left|Q_{\nu_{n+1}}^{c\,k(i)}\right|}\right)$$

$$\dot{\gamma} \geq 0$$

$$\dot{\gamma}\Upsilon = 0$$

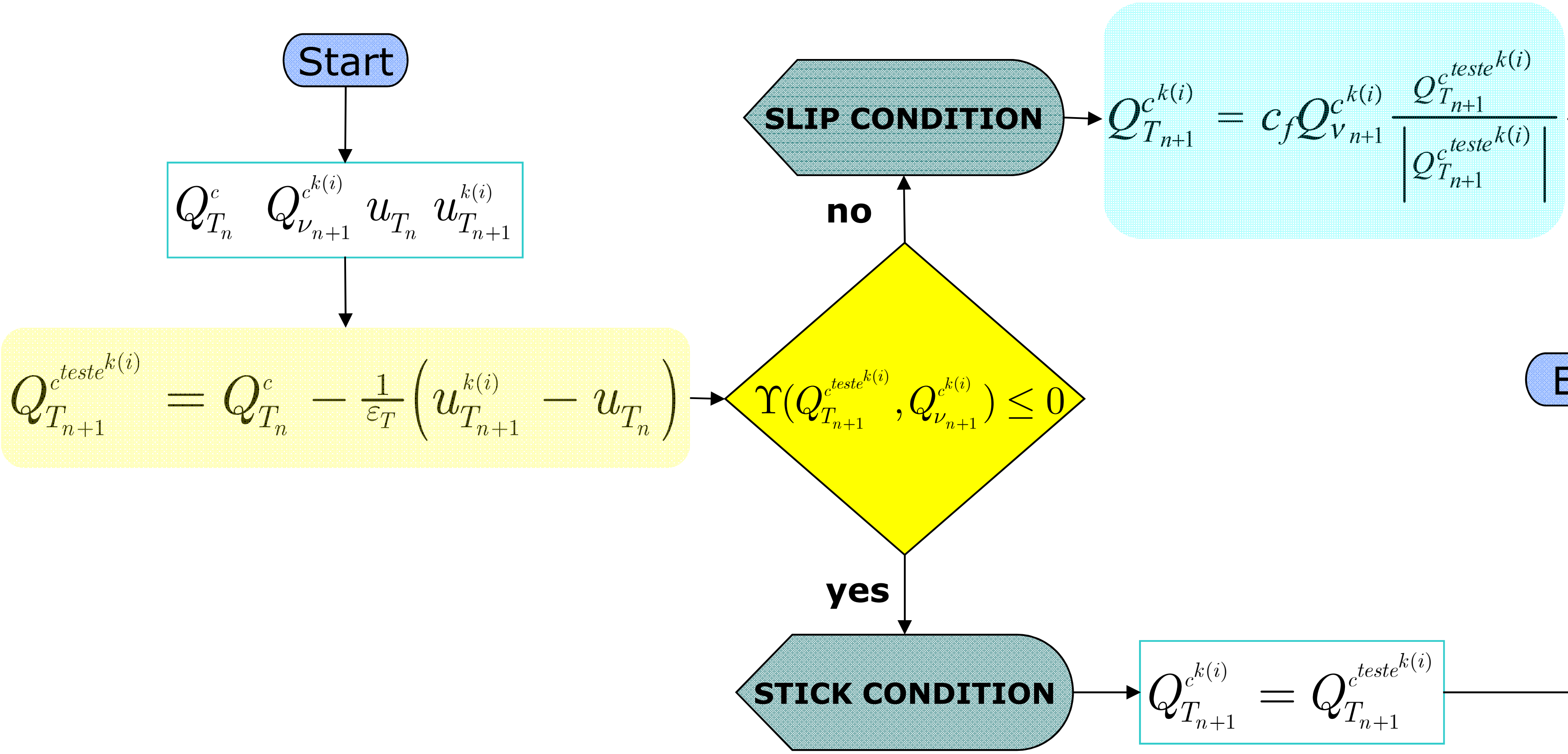


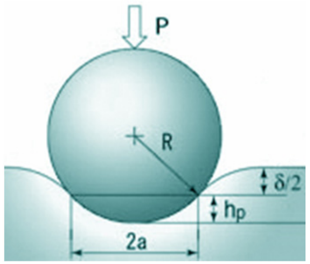
- *Stick condition* $\Upsilon \leq 0$
- *Slip condition* $\Upsilon > 0$



Unilateral Contact with Friction

- **Frictional Algorithm**



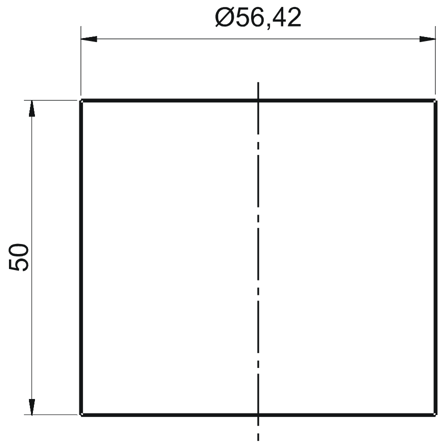


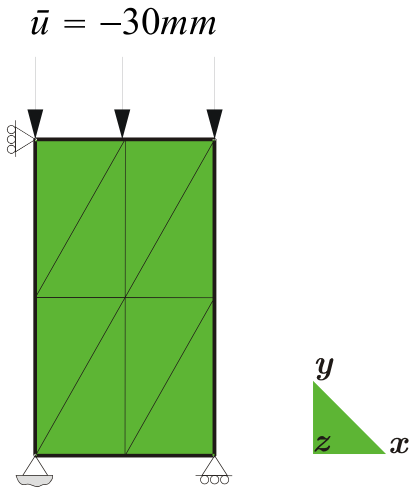
Unilateral Contact with Friction

- **Examples using EFG**
  - Uniaxial Compression (with the same d.o.f of the FEM example)

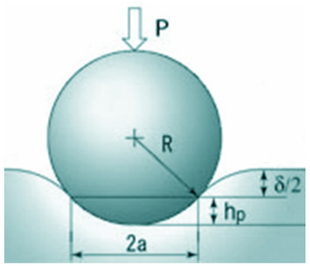
$$tol_1 = 10^{-6} \quad s = 1,5 \quad 100\,load\ \ steps$$

$$\varepsilon_u = 10^{-6} \quad 7\,pintg$$



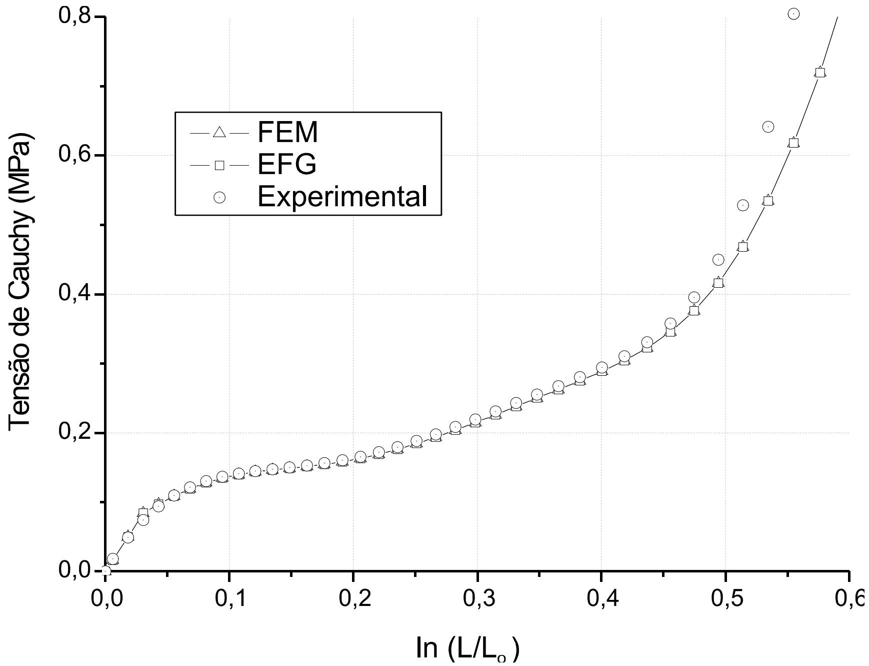


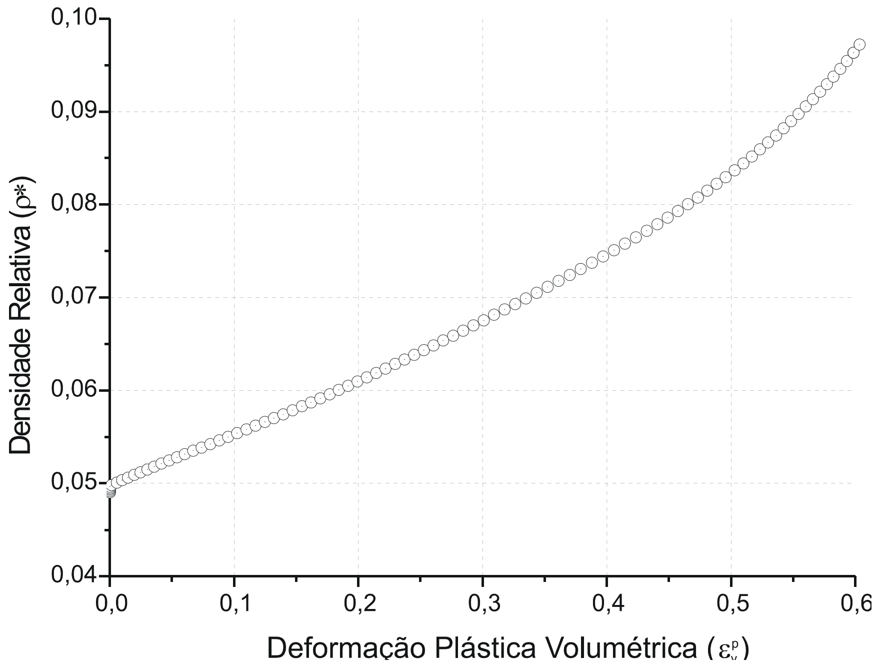
$$\tau_y^o = 0,082034\ Mpa$$
$$p_c^o = 0,040470\ MPa$$
$$\varsigma = 0,50$$
$$E_m = 928,09288\ MPa$$
$$\rho_o^* = 0,049$$
$$\nu_p = 0,00$$
$$\nu = 0,25$$
$$\gamma = 1,54$$
$$c = 0,30.$$



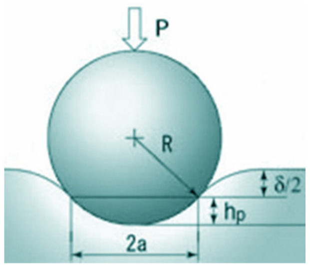
Unilateral Contact with Friction

- Uniaxial compression and the densification process





ZHANG, J. *at al* (1998)



Unilateral Contact with Friction

- Axisymmetric Cone Slice
  (with the same d.o.f of the FEM example)

$1000\, load \;\; steps$

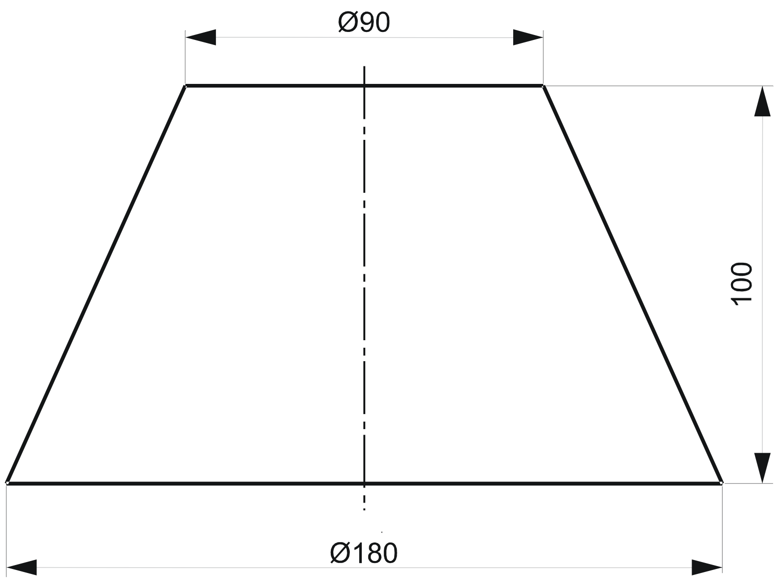
$7\, pintg$

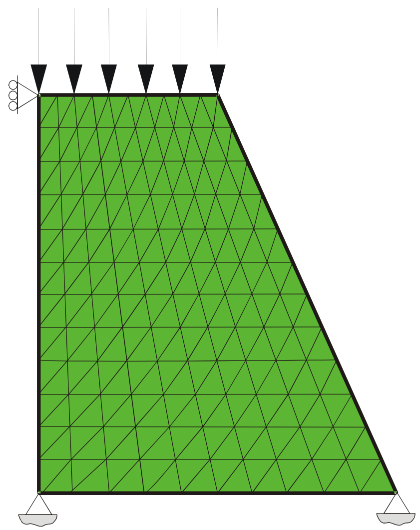
$tol_1 = 10^{-6}$

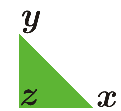
$\overline{u} = -80mm$

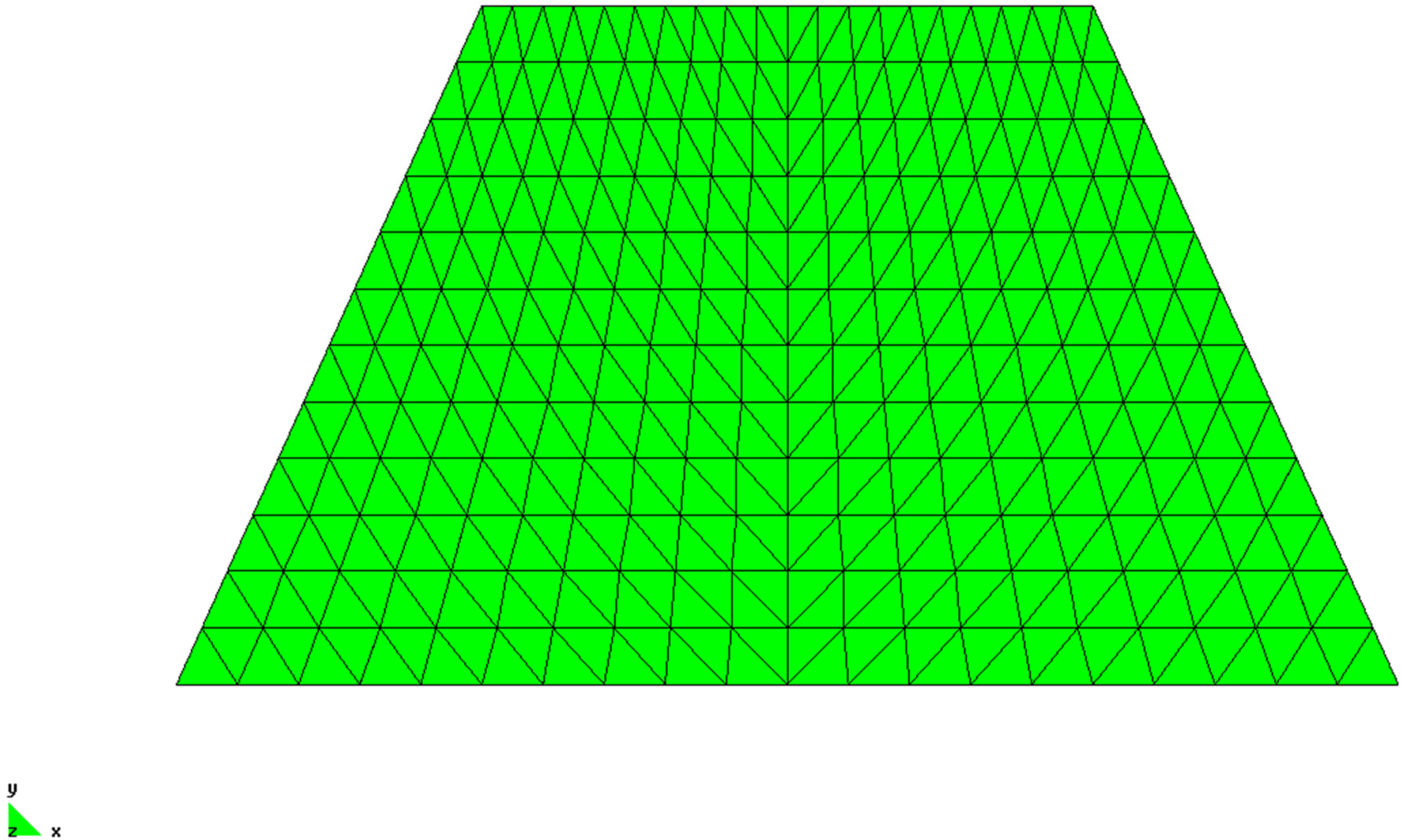
$s = 1,5$

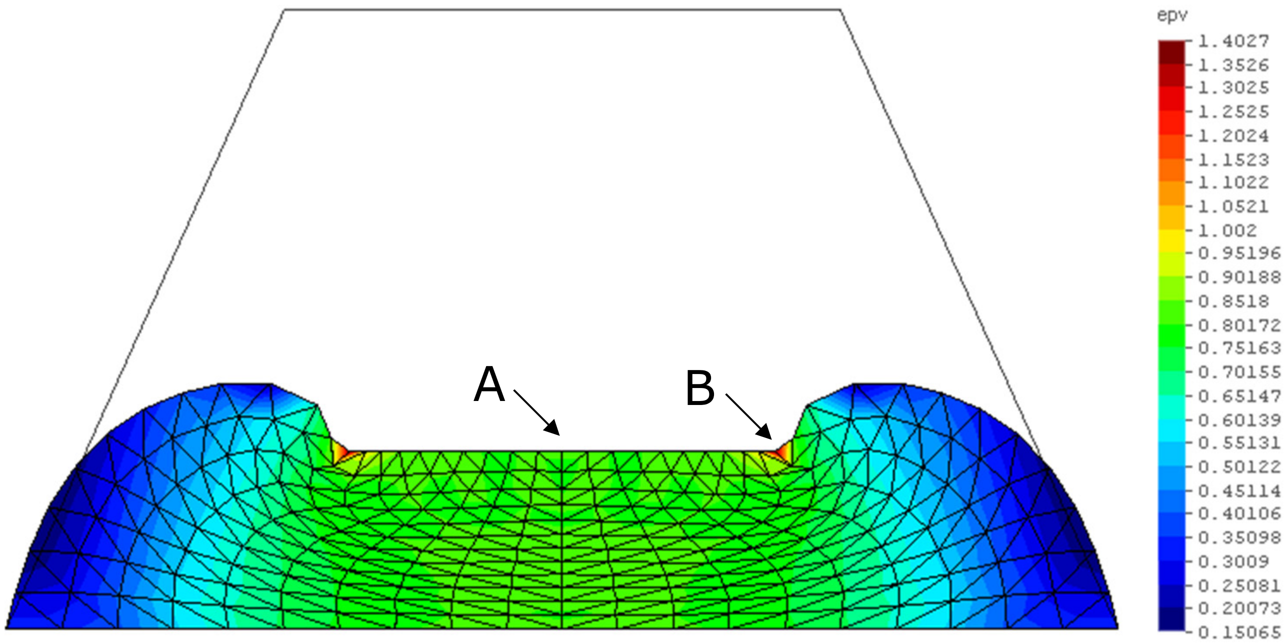
$\varepsilon_u = 10^{-6}$



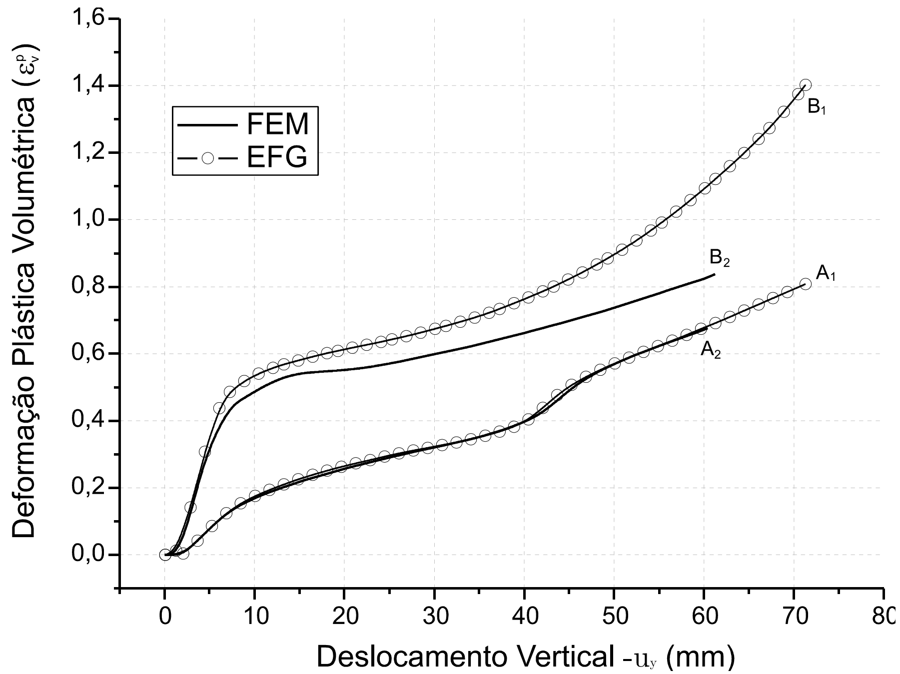





y
z
x

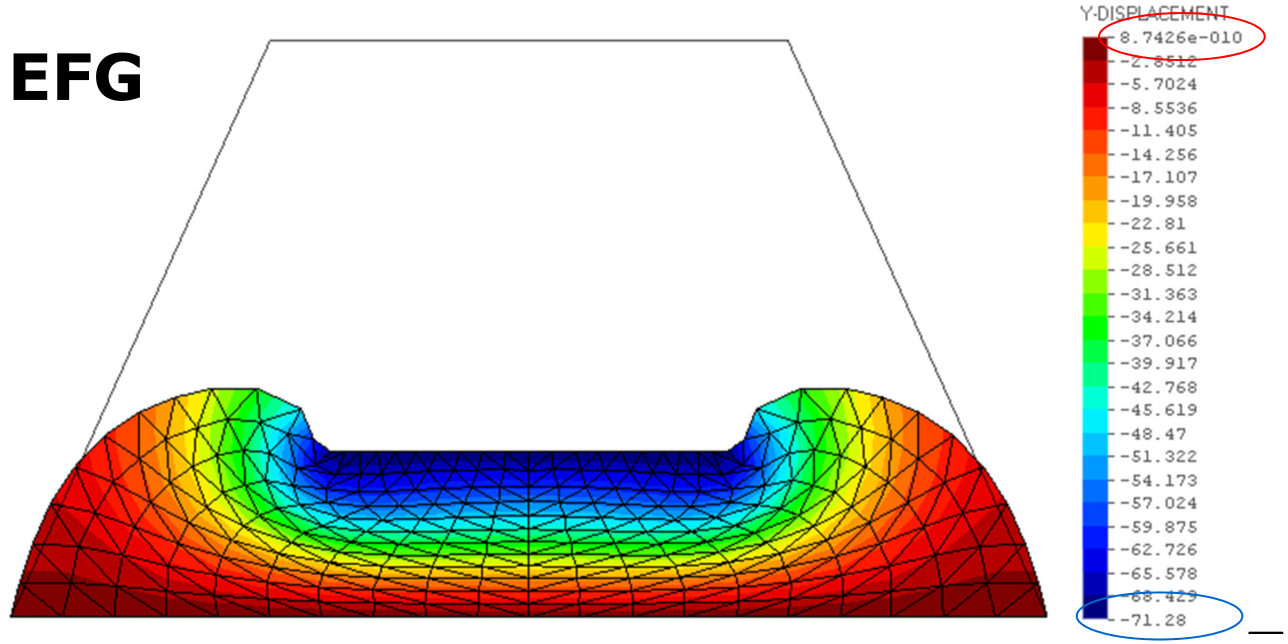


A
B
epv
1.4027
1.3526
1.3025
1.2525
1.2024
1.1523
1.1022
1.0521
1.002
0.95196
0.90188
0.8518
0.80172
0.75163
0.70155
0.65147
0.60139
0.55131
0.50122
0.45114
0.40106
0.35098
0.3009
0.25081
0.20073
0.15065



Deformação Plástica Volumétrica (ε_v^p)
FEM
EFG
B_1
B_2
A_1
A_2
1,6
1,4
1,2
1,0
0,8
0,6
0,4
0,2
0,0
0
10
20
30
40
50
60
70
80
Deslocamento Vertical $-u_y$ (mm)

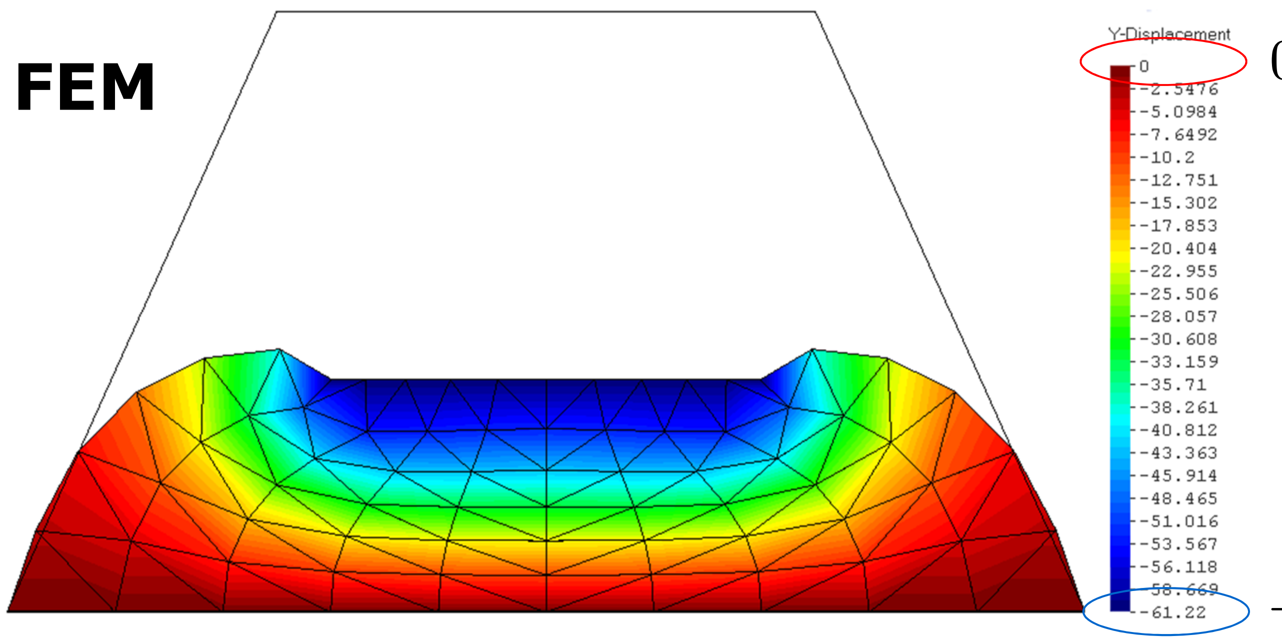
EFG



$8,74.10^{-10}$

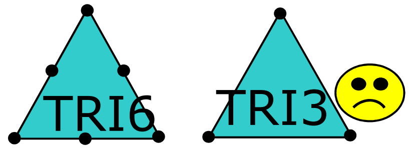
$-71,28 \quad (15\%) \gg$

FEM

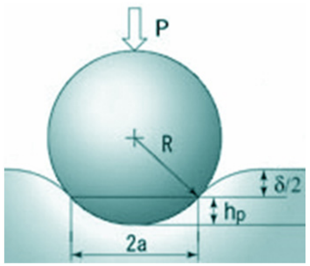


$0,00$

$-61,22$



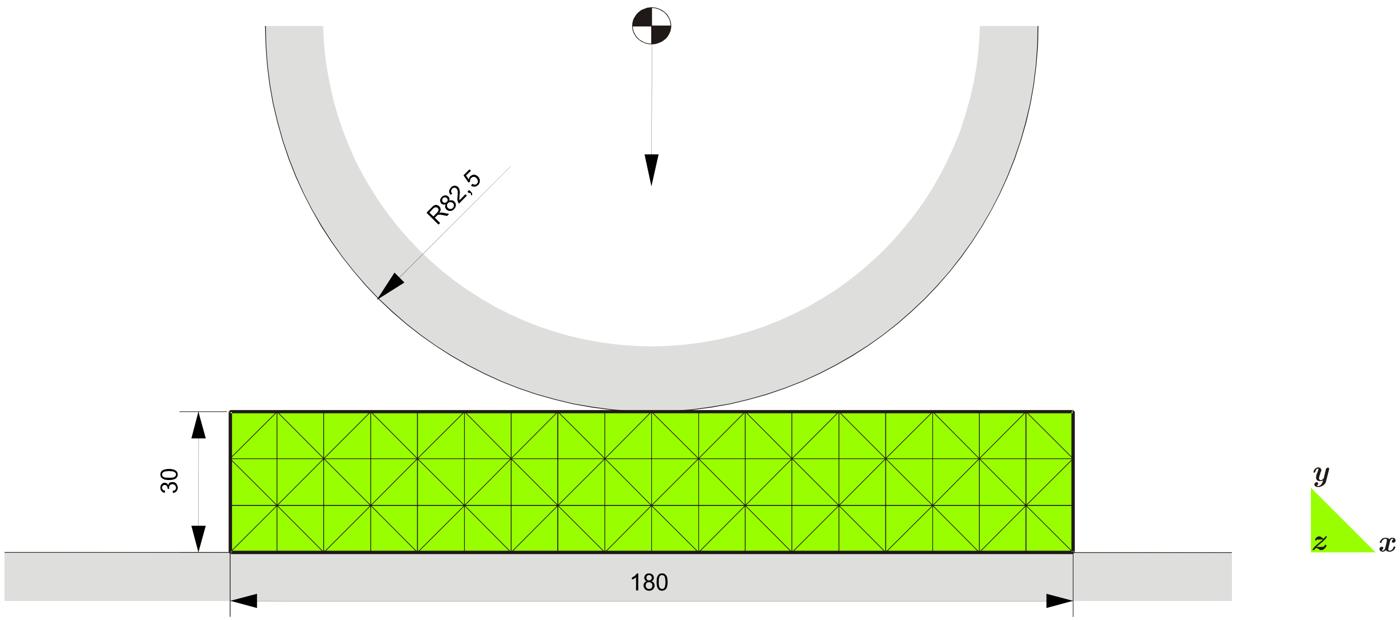
Note: Calculated with TRI6 visualized as TRI3 element!



Unilateral Contact with Friction

- Indentation Test

$$\overline{u} = -15mm$$



$$tol_1 = 10^{-6}$$

$$tol_2 = 10^{-6}$$

$$s = 1,5$$

$$c_f = 0,10$$

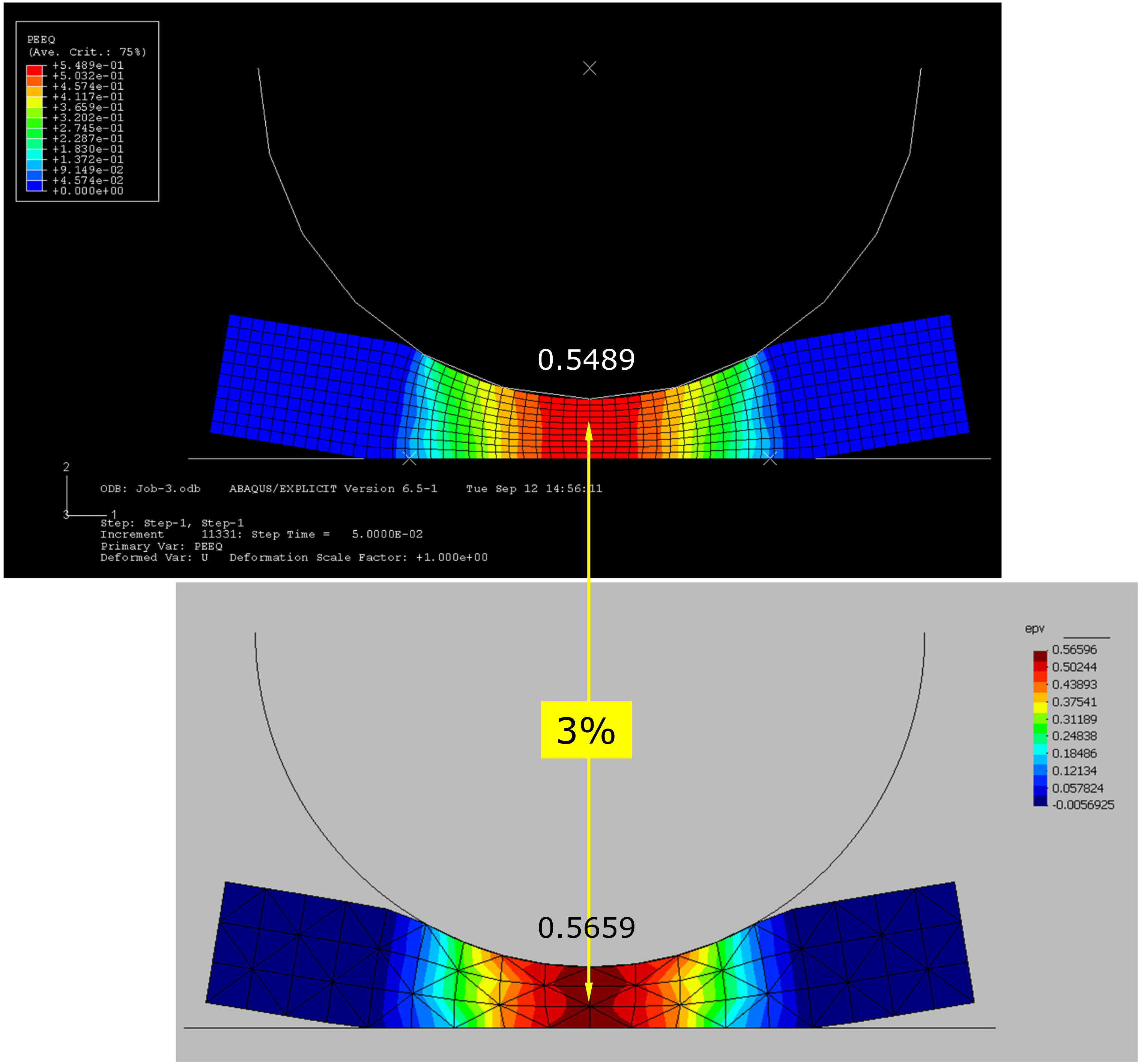
$$\varepsilon_\nu = 10^{-4}$$

$$\varepsilon_T = 10^{-3}$$

$$\varepsilon_u = 10^{-6}$$

$$1000\ steps$$

$$7\ pintg$$


PEEQ
(Ave. Crit.: 75%)
+5.489e-01
+5.032e-01
+4.574e-01
+4.117e-01
+3.659e-01
+3.202e-01
+2.745e-01
+2.287e-01
+1.830e-01
+1.372e-01
+9.149e-02
+4.574e-02
+0.000e+00
0.5489
2
1
3
ODB: Job-3.odb ABAQUS/EXPLICIT Version 6.5-1 Tue Sep 12 14:56:11
Step: Step-1, Step-1
Increment 11331: Step Time = 5.0000E-02
Primary Var: PEEQ
Deformed Var: U Deformation Scale Factor: +1.000e+00
3%
epv
0.56596
0.50244
0.43893
0.37541
0.31189
0.24838
0.18486
0.12134
0.057824
-0.0056925
0.5659

Conclusions

- **The constitutive model**

  - The features
    - a single-surface yield criteria;
    - a non-associated plastic flow law;
    - the relative density dependence $\mathbb{D}(\rho^*)$;

  - Show a good prediction for the responses of rigid polymeric foams under simple and complex monotonic loading conditions.

  - The yield surface parameters are simple and can be obtained from two independent experiments
    - Uniaxial compression (ISO 844 / ASTM D1621-04a);
    - Hydrostatic compression;

  - The relative density dependence has as a consequence a correction in the tangent modulus.

$$\frac{\partial \bar{\boldsymbol{\tau}}}{\partial \mathbf{F}} = \left(\frac{\partial \mathbb{D}(\rho^*)}{\partial \mathbf{F}} \mathbf{E}^{\mathrm{e}} \right) + \left(\mathbb{D}(\rho^*) \frac{\partial \mathbf{E}^{\mathrm{e}}}{\partial \mathbf{F}} \right)$$

Conclusions

- **EFG x FEM**

  - EFG method showed to be:
    - Able to withstand the analysis of very large deformation processes, without remeshing and breaking up;
    - More robust to capture high deformation levels and deformation gradients;
    - More expensive with respect the computational aspect (more integration points);
    - In another hand, load step size bigger than the FEM. (small number of interactions!)

  - The choice of the penalties values (in both cases EBC and Contact) implies in changes at the convergence rate;
    - Difficulties to find the “right” value for each problem.

Conclusions

- **Numerical Aspects**

  - The *Returning Map Algorithm* do not capture *critical points* (limit or path bifurcation).
    - For this reason, strategies was performed to improve the *condition number* of the local tangent modulus at the plateaus. (matrix is singular if the condition number is infinite)
  - Compression causes stress bottom up due to foam consolidation.
    - Problems to insure convergence (*Turning* causes low convergence rates);
    - Further, the hardening laws are interpolated by a polynomial fitting. After 60% of the logarithm strain, this fitting don't represent the experimental data.
    - For this reason, the divergence between the numerical and experimental data is plausibly.
  - At overall, the numerical procedure showed to be adequate to describes the rough non-linearities (material and contact with friction).